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Algebra discourses in mathematics and physics textbooks: comparing the use of algebraic symbols

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ABSTRACT

It is agreed that algebra has an important role in physics, particularly through handling symbols. A lot of previous research has focused on how mathematics is used in physics from perspectives where mathematics is taken for granted, and not addressing potential differences of mathematics in the physics classroom and in the mathematics classroom. Studies addressing differences between both subjects have been based on researchers' own experiences of mathematics in both subjects. Thus, more focused empirical research is needed. The purpose of this study is to clarify similarities and differences between mathematics and physics concerning the use of algebraic symbols. Analyses were based on comparisons between upper secondary textbooks in mathematics and in physics from a discourse perspective. Statistical methods were used to decide if there were any significant differences between the subjects. Results showed an overlap in the algebra discourse in both subjects, but also several differences concerning core aspects of algebra. For example, a higher number of different algebraic symbols in equations in physics than in mathematics, and algebraic symbols are more seldom referred to by words in mathematics than in physics. This can make it difficult for students to identify similarities in the algebraic discourses in both subjects.

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1. Introduction

Mathematics as a subject has somewhat of a special position in relation to other subjects, since it is a requirement for studies of several other major subjects. In particular, physics is usually seen as depending much on mathematics, where mathematics is commonly said to be the language of physics, and knowledge of mathematics is seen as a prerequisite for being able to learn physics (e.g. Bing & Redish, 2009; Hansson et al., 2015, 2021; Nilsen et al., 2013; Torigoe & Gladding, 2011). However, the more detailed relationships between mathematics and physics education are not well understood. For example, knowledge about supportive use of mathematics in physics education has been described as fragmentary (Uhden et al., 2012). In addition, research has not succeeded in clarifying the major

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difficulties many students have with the use of mathematics in physics, for example when handling and interpreting algebraic symbols in the physics context (Redish & Kuo, 2015). One reason for this lack of progress in research on relationships between mathematics and physics can be that a certain perspective has been dominating, where knowledge of mathematics is seen as being *transferred* by the students from the mathematics classroom to the physics classroom. We might need to see the relationship as more complex than that, for example, by acknowledging that *mathematics in physics* can be something different than *mathematics in mathematics* (Redish & Kuo, 2015).

In this study, we acknowledge the complexity in the relationships between mathematics and physics, and by taking a discourse perspective in comparing Swedish textbooks in physics and mathematics, we make a contribution to elucidate this relationship. We delimit the comparison to the domain of algebra, since algebra is often described as a problematic area, both in mathematics and in physics education (e.g. Carraher & Schliemann, 2007; Nilsen et al., 2013).

In addition to a more theoretical contribution to elucidating the relationship between mathematics and physics, comparisons between the subjects are necessary in order to provide valuable insights into how teaching can be organised (Heck & van Buuren, 2019). If there are similarities between the subjects, it may be reasonable for teaching and for students' learning to focus on whether and how there is some kind of transfer of knowledge from the mathematics classroom to the physics classroom. And if there is no such transfer, it may be relevant to consider why students do not see these similarities and do not use existing knowledge from the subject of mathematics when they study physics. If there are differences between the subjects, that is, if the algebra in the physics classroom is not exactly the same as the algebra used in the mathematics classroom, this can highlight questions for teachers and textbook authors about if and how the teaching of mathematics can be made more useful for physics and how the teaching of physics can be organised so that it does not assume that all mathematics is well known to students. Comparisons between the subjects can also show more specifically where and how there are similarities and differences, for example so that the handling of algebra in physics teaching can be adapted accordingly.

2. Background

2.1. Mathematics and physics

Much research on the relationship between mathematics and physics has focused on students, often concerning problems associated with lack of knowledge or lack of transfer of knowledge. For example, Nilsen et al. (2013) summarise many studies that address reasons why students struggle with mathematics in physics. These studies either focus on students' lack of mathematical knowledge or that they do not know how to apply their knowledge in physics. However, it is limiting to reduce the relationship between mathematics and physics to issues of transfer of knowledge. Analyses of the relationship from historical and philosophical perspectives show, for example, deep interrelations between mathematics and physics, where mathematics cannot be reduced to a tool for calculations in physics (Uhden et al., 2012).

Furthermore, the focus on issues of transfer of knowledge from mathematics to physics does not acknowledge that the mathematics in physics classrooms is not necessarily the same as mathematics in mathematics classrooms. Instead, these can be seen as two different languages (Redish, 2006; Redish & Kuo, 2015). Planinic et al. (2012) showed that students have trouble with identifying similar tasks in mathematics and physics, and that they succeed on tasks in mathematics, while failing on corresponding tasks in physics, for example, on multiple choice tasks referring to the concept of line graph slope. In physics, students were presented with a velocity-time graph of an object's motion and in mathematics a *y*-*x* graph, both including a line with negative slope, and students were asked to choose the correct statement about the situation in both contexts. This is thought to relate to the difficulties of interpreting mathematics in the context of physics and that mathematics in physics classrooms can be different from mathematics in mathematics classrooms (Planinic et al., 2012). Karam et al. (2019) addressed a similar phenomenon when analysing historical genesis of differences between mathematics and physics, and didactical implications of such differences. A main conclusion was that it is not fruitful to claim that students should have learned mathematics needed for physics in mathematics courses.

Research that directly addresses potential differences between the subjects of mathematics and physics is scarce (cf. Redish & Kuo, 2015), but there is plenty of research that focuses on how mathematics is used in physics. The following four strands of research focus on physics, and analyses are done, empirically and/or theoretically, concerning how different aspects of mathematics relate to, or are part of, physics.

One strand analyses mathematical competencies that are needed to solve physics tasks. A common conclusion from different studies is that many physics tasks require mathematical competencies (Author, 2016a; Nilsen et al., 2013) in different ways: The need for creative mathematical reasoning, that is, to use known mathematics in new situations, is prominent (Author, 2016a, 2017). This is also the case for the handling of symbols (i.e. interpret formal mathematical language and transform everyday language to formal mathematical language, as well as handle and manipulate formulas and equations), as well as the handling of mathematical representations (i.e. use and shift between symbols, graphs, diagrams etc.) and mathematical modelling (i.e. describing the physical word mathematically by deriving relationships between variables, e.g. a formula) (Nilsen et al., 2013).

A second strand focuses specifically on aspects of modelling, where different perspectives are described concerning how mathematics relates to the physical world (Redish, 2006), to reality and theoretical models (Hansson et al., 2015) and to physical models (Uhden et al., 2012). Such perspectives can be used to, for example, analyse how students relate to or use mathematics during physics lessons.

A third strand distinguishes between 'technical' and 'structural' roles of mathematics to highlight the use of mathematics in physics. Karam (2014) uses this distinction to analyse the teaching strategies of a lecturer, while Hansson et al. (2015) analyse the communication between students and teachers.

Finally, Redish and Kuo (2015) describe a fourth strand as a combination of a lens from cognitive linguistics and aspects of 'resources' as a perspective on the use of mathematics in physics. They use this perspective to analyse students' reasoning around equations in physics.

In summary, common for the research above is a focus on mathematics in physics, where the mathematics is taken for granted, and any differences between the two subjects are not taken into consideration. In particular, we have not found any studies that perform a structured empirical analysis of comparisons between mathematics and physics. Thus, research has so far produced a somewhat fragmented picture of the relationship between mathematics and physics, especially concerning students' experiences regarding this relationship. In particular, we do not know if and how students meet 'a different mathematics' in the physics classroom compared to the mathematics classroom. Our study contributes by empirically examining similarities and differences between textbooks in mathematics and physics from a discourse perspective, focusing on aspects of algebra.

2.2. Algebra in mathematics and physics

Algebra has an important role in physics. For example, it is common that physics tasks require the use of algebra, in particular through symbol handling (as accounted for above, Nilsen et al., 2013). Thus, the role of algebra in physics is an important area for research, and previous research that has focused on this is summarised here.

Studies show that physics tasks that contain algebra are more difficult for students, compared with other physics tasks. For example, university students fail more often on physics tasks that contain symbols, and relations between symbols, for particular quantities, than on similar tasks where numerical values are given instead of the symbols (Torigoe & Gladding, 2011). This is thought to depend on difficulties in understanding the meaning of the symbols, since very few of the errors are due to manipulation errors of the equations that include the symbols. Upper secondary students also express that it is physics tasks that contain manipulations with symbols for quantities that they fail the most on (Angell et al., 2004). It has therefore been suggested that low results on the physics test in TIMSS Advanced can depend on students' limited algebra knowledge (Nilsen et al., 2013).

As discussed above concerning relationships between mathematics and physics in general, the focus on students' lack of knowledge can reflect a focus on issues of transfer of algebra knowledge from mathematics courses. However, the algebra used and needed in physics may differ from the algebra in mathematics, since 'although the formal mathematical syntax may be the same across the disciplines of mathematics and physics, the uses and meanings of that formal syntax may differ dramatically between the two disciplines' (Redish & Kuo, 2015, p. 562). The following researchers discuss differences between mathematics and physics regarding aspects of algebra, although their discussions are not based on any structured empirical analyses of mathematics and physics. With respect to complexity and the number of different variables, Heck and van Buuren (2019) note that variables often occur in isolation in mathematics (e.g. in a quadratic expression, $x^2 + 2x + 3$), while in physics, focus often is on relationships between several variables (e.g. $U = R^*I$). Similarly, Redish (2006) notes that very few equations in calculus contain more than one symbol (incl. symbols for variables, constants, parameters etc.), while many different symbols are usually used in physics, from three to six symbols for different variables, constants, parameters etc., or more in one equation. Furthermore, it is noticed that symbols used as representations for variables differ between the subjects, for example, the Latin letters x, y, z or t are usually used in calculus while many other symbols (e.g. both various Latin and Greek letters) are used in physics (Heck & van Buuren, 2019; Redish, 2006). In particular, meaning is loaded to symbols in physics through the choice of symbols depending on the context (e.g. m, V or E). Loading meaning to symbols is further discussed by Ellermeijer and Heck (2002) and Redish and Kuo (2015). Regarding ambiguous use of notation, Ellermeijer and Heck (2002) also note that a function, a sample of function values and a single function value are mixed up in the symbolic writing in physics, while this rarely occurs in mathematics.

In summary, common for the studies that address differences between mathematics and physics concerning aspects of algebra is that they do not rely on any structured empirical data and analyses of such data. Instead, examples of differences are based on the researchers' own perspectives and experiences of mathematics and physics. Thus, there is a clear need for more focussed empirical research on this issue. Our study contributes by empirically examining similarities and differences between textbooks in mathematics and physics, focusing on aspects of algebra.

3. Purpose and research questions

The purpose of this study is to clarify similarities and differences between mathematics and physics concerning the use of algebraic symbols. We focus on symbolic sequences that include algebraic symbols and perform discursive comparisons between textbooks in mathematics and physics. We will answer the following three-part research question (RQ).

What similarities and differences are there between textbooks in mathematics and physics, concerning:

1. How are the symbolic sequences constructed, concerning size and complexity?
2. What are the characteristics of the symbolic sequences?
3. What types of words are used to label or name symbolic sequences, or parts of sequences?

Central concepts such as algebraic symbols and symbolic sequences are defined in the Theoretical perspective section below, and summarised in Table 1. By answering this three-part RQ, we can contribute with empirical bases in relation to different statements or descriptions from previous research that have been described above. RQ1 relates to descriptions of differences in how symbolic sequences are constructed in mathematics and physics. Beyond the mere syntax, previous research has also described more general aspects of differences between mathematics and physics concerning the use of symbols, and RQ2 addresses this issue by focusing on an overall characterisation of the symbolic sequences that are used. Finally, previous research has described that meaning is loaded to symbols in physics more often than in mathematics, and RQ3 highlights one perspective on this issue by examining if and how different words are used when addressing symbols.

4. Theoretical perspectives

In this study, we compare textbooks in mathematics and physics, which can be seen as a tool for comparing different discourses in educational settings. In particular, the purpose of our study is to examine if and how there are different algebraic discourses in mathematics and physics education. Here, we position our study in relation to the concepts of discourse and algebraic discourse.

A subject, or a part of a subject, is considered *a discourse* that is constituted by four different characteristics: *word-use*, *visual mediators*, *routines* and *endorsed narratives* (Sfard, 2008). For a discourse to be considered mathematical, the communication should be about mathematical concepts, such as five, variable, symmetry etc. Word-use refers to mathematical vocabulary unique for a particular discourse, such as variable or function, but also to colloquial words that are used with particular meanings, for example 'balance scale' to explain the equal sign or 'growing' in relation to positive slope. Visual mediators are the visible objects that are developed for mathematical communication, for example graphs and symbols. Routines are the rules that describe repetitive patterns that are typical for the particular discourse, for example specific procedures for solving equations. Endorsed narratives refer to statements that are agreed to be true in the particular discourse, such as theorems and proofs, which describe objects, relations between objects, or processes (e.g. Tabach & Nachlieli, 2016). Participating in a particular discourse implies that these four characteristics are used in similar ways by all interlocutors (Sfard, 2008). At the same time, 'thinking is defined as the individualized version of (interpersonal) communication' (Sfard, 2008, p. 81), thus, thinking and speaking are inseparable within this theoretical perspective. This implies that discourse, developed through interaction as a means of communication, is also seen as individual. Our study does not attempt to perform any complete characterisation of discourses, but focuses on algebraic symbols, as a type of visual mediator, and on the word-use in relation to the symbols. Word-use is delimited to the research question about the words that are used to name or label symbols.

In this study, the focus is on the *algebraic discourse*, here defined as a discourse dealing with symbolically represented indeterminate quantities and relations between them in an analytical way (Radford, 2018). Indeterminate quantities imply that more than given numbers or other mathematical entities are involved in algebraic situations. These 'quantities can be unknowns, variables, parameters, generalized numbers, etc.' (Radford, 2018, p. 8). The analytical dealing with indeterminate quantities implies that, despite being unknown, the quantities are operated on as if they were known. For example, students who deduce a solution through logical reasoning based on an assumption of equality, are dealing analytically with the equation, as opposed to students using trial and error (Radford, 2018). Besides the symbolically represented indeterminate quantities, we include non-numerical symbols for constants (e.g. π or c (speed of light)), and label this group of symbols *algebraic objects*, where object refers to the smallest algebraic entity (i.e. x, V, g or v_y, and not $x + y$, or mg since these include operations on entities). In our analysis, we focus on the use of *algebraic symbols*, that is, all symbols that are not numerals, but letters or other signs that represent some number (i.e. algebraic object), or operation involving such numbers or relation between such numbers. For example, as stated above, v_y is one algebraic object,

Table 1. Definitions of key concepts used in the study.

Concept	Definition/description
Indeterminate quantities	Imply that more than given numbers or other mathematical entities are involved in algebraic situations (e.g. unknowns, variables, parameters, generalised numbers).
Algebraic object	The smallest non-numerical symbolically represented indeterminate quantity, incl. symbolically represented constants (e.g. G (gravitational constant) or π (pi))
Algebraic symbols	All symbols that are not numerals, but letters or other signs that represent algebraic objects, or operations involving algebraic objects or relations between algebraic objects
Symbolic sequence	An unbroken sequence of mathematical symbols including at least one algebraic object (i.e. can include equal signs, e.g. $3 + 4 = 3x + 4b$ (equation) or $f(x) = 3x + 4b$ (formula)).
Algebraic discourse	Discourse dealing with symbolically represented indeterminate quantities and relations between them in an analytical way

and this object is represented by a specific combination of two algebraic symbols v and y. Table 1 provides an overview of definitions for different key concepts that are used in this study.

5. Method

5.1. Data

In order to empirically examine differences and similarities between the handling of algebraic symbols in physics education and in mathematics education, this study focuses on upper secondary students' textbooks. It is well known that textbooks are a primary resource for teachers' lesson planning and students' task practicing, both in mathematics (e.g. Fan et al., 2018) and in physics (e.g. Banilower et al., 2013). The focus in this study is not on teachers' or students' actual use of mathematics, but on the algebraic discourse students meet in the different subjects. Thus, delimiting empirical differences and similarities to the content of textbooks is justified. An additional delimitation is to only consider textbooks for the first upper secondary course in mathematics and in physics. For upper secondary school in Sweden, the subject of mathematics is divided in different tracks (a, b and c) depending on the programme. Students in theoretical programmes with direction towards science and technology take the c-track in mathematics. Each track is furthermore divided in succeeding courses, Mathematics 1c, Mathematics 2c and Mathematics 3c. In a similar way, the subject of physics is divided in two succeeding courses, Physics 1 and Physics 2, where Physics 2 is optional for technology students. Usually, Mathematics 1c and Physics 1 are taught in parallel from the beginning of year one in upper secondary school, but sometimes slightly offset so that students have studied some parts of the Mathematics 1c course before the Physics 1 course starts.

There is no record of sales statistics from publishers of textbooks for Swedish schools. Therefore, the choice of textbooks was based on a previously conducted survey of the most commonly used textbooks in mathematics (Author, 2016b). After having identified a commonly used textbook for Mathematics 1c, the textbook for Physics 1 was chosen from the same publisher. Based on this, the following books were analysed in this study: *Exponent 1c* (Gennow et al., 2017) for mathematics and *Impuls Fysik 1* (Fraenkel et al., 2011) for physics.

The physics book contained 450 pages divided into 11 chapters, each chapter focusing on a specific physics topic. Because different physics topics could include different aspects of an algebraic discourse, it was decided to include various topics in the analysis. Furthermore, it is assumed that the greatest variation of a discourse occurs when a topic is introduced, and variation is less prominent during continued use and repetition. Thus, in order to get a representative sample of the algebraic discourse in the physics book, it was decided to include parts containing theory texts and solved examples, and exclude summary pages and practicing tasks. To capture various uses of algebraic symbols in physics it was furthermore decided to not include parts in the physics textbooks that focused on physics conceptualisation that did not relate to the use of mathematics. Thus, parts including contextual and conceptual discussions without mathematical anchoring, as well as historical and facts pages were excluded from the analysis. Of the 240 pages that contain theory texts and solved examples, 25 pages were randomly chosen to be analysed.

If the chosen page included theory text of a subsection, or an example, that began on a previous page or continued on a succeeding page, this text was included in the analysis. If the page consisted of both theory text and an example, the part covering the most of the page was included in the analysis. For example, if the first four lines of the page belonged to theory and the rest of the page contained an example of a solved task, only the example was analysed.

The mathematics book contained 340 pages divided into 6 chapters, each chapter focusing on a specific mathematical content divided into different topics treated in subchapters. In order to capture the handling of algebraic symbols (i.e. the algebraic discourse) in mathematics, in situations that could be considered relevant in comparison with the situations in physics, it was decided to include the following chapters and subchapters in the mathematics book in the analysis: Algebra, Trigonometry (incl. Pythagorean theorem and vectors), Scientific applications and Functions. In the same way as for physics, it is assumed that the greatest variation of a discourse occurs when content is introduced. Thus, the included pages were delimited to the ones that contained theory texts and solved examples. Content in the mathematics book that was excluded are Number sense, Geometry (with respect to repetition of geometrical figures and their properties, and of Cartesian coordinate system, as well as introduction to mathematical proofs of geometrical theorems), Percentage, and Probability and statistics. Although parts of these excluded chapters might include some algebraic symbols, these were considered to be so few that it would not contribute to reflecting the algebraic discourse in mathematics in a relevant manner in relation to the use of algebra in physics. Similar as for the physics book, parts that included contextual and conceptual discussions, as well as historical and facts pages, were excluded from the analysis. This is because, at a first glance, these sections usually did not include any mathematical symbols (i.e. likely even fewer algebraic symbols) and are thus not considered to contribute to a representative sample of the algebraic discourse in the mathematics book. In total, approximately 55 pages, of the mathematics book's 340 pages, were included in the analysis.

Somewhat different methods for selection of data were deliberately used for mathematics and physics. In mathematics, the purpose was to capture certain types of content, concerning the handling of algebraic symbols, and thereby obtain the most representative sample of the algebraic discourse, suitable for comparisons with physics. Therefore, specific complete sections in the mathematics textbook were selected, which resulted in a surmountable number of pages. In physics, the purpose was to get a representative sample of the algebraic discourse in the subject, from a large number of possible pages. Therefore, a random sample was used, which resulted in enough units of analysis for statistical analyses.

5.2. Analysis procedure

A unit of analysis was delimited to an unbroken sequence of mathematical symbols including at least one algebraic object visualised by one or more algebraic symbols. Other mathematical symbols in the sequence could be the equal sign, mathematical operations, numerals, implication arrows, indices and similar. The sequence is considered broken if there is a punctuation mark after a symbol, or if there is a new row without explicit marking that the sequence of mathematical symbols continues on the new row, as well as if there is a written word after a symbol. These units of analysis will be referred to as *symbolic sequences*. This allows a symbolic sequence to be anything between very short (e.g. x or $E_1 + E_2$) and very long (e.g. $W = mg(h_2\text{-}h_1) = mgh_2\text{-}mgh_1 = E_{p2}\text{-}E_{p1}$). In total, 1014 symbolic sequences from the 55 pages in the mathematics textbook, and 296 symbolic sequences from the 25 pages in the physics textbook, were included in the analysis.

To answer our research questions, each unit of analysis was examined with respect to different aspects. In order to empirically reveal the construction of symbolic sequences (RQ1), the following aspects were used to capture the complexity or size of sequences: total number of algebraic objects and total number of different algebraic objects; total number of equal signs; total number of algebraic objects and total number of different algebraic objects in sequences with more than one equal sign; total number of mathematical operations and total number of different mathematical operations, as well as which operations; and also the use of special types of symbols by examining the total number of algebraic objects that included an index (e.g. b_2, e_x), that were preceded by capital delta (e.g. Δh, Δt_b) and that included Greek letters (except Δ). These special types of symbols are known, from experience, to occur frequently in physics, and therefore it is relevant to empirically determine if they are more common in physics than in mathematics. Operations are here used with a broader meaning than what is formally counted as mathematical operations. It is, for example, distinguished between addition signified by the plus symbol and addition signified by the summation symbol. It is also distinguished between different situations that include the division sign, where the operation 'division' relates to all situations where there are no variables in the denominator, and the operation 'inverse proportionality' relates to situations with a number in the numerator and a variable in the denominator. The above aspects are chosen to capture fundamental aspects of the construction of symbolic sequences, but they also relate to some specific statements in previous research about symbols in mathematics and physics (e.g. the number of used symbols, Heck & van Buuren, 2019).

As a characterisation of symbolic sequences (RQ2), each unit of analysis was categorised according to the dominating overall property, where categories were created bottom-up, that is, based on all situations occurring in the data. This resulted in the five categories presented in Table 2. The categories were treated as disjoint, to best capture the dominating overall property of how each symbolic sequence was used, and each analysed symbolic sequence only belonged to one category.

Furthermore, for each symbolic sequence, it was denoted with 'yes' or 'no' if the sequence was a part of a preceding unit of analysis. This allowed to capture and compare the existence of relationships between symbolic sequences in mathematics and in physics. In the following example,

Table 2. Categories of the overall properties for the symbolic sequences.

Categories of overall properties	Examples
1: The sequence was a single algebraic object	$x, V, b_2, \Delta h$
2: The sequence was a single mathematical expression	$3v, (k_1 + k_2)u, h \cdot A, mc^2$
3: The sequence was a single mathematical relation, such as an equation or a formula, incl. derived relations	$x + 9 = 11, W = F{\cdot}\Delta s, O = \pi{\cdot}d,$ $a + b(c + d) = a + (b{\cdot}c + b{\cdot}d) = a + bc + bd$
4: The sequence specified values for the quantities	$u = (2,-1),$ $h = 39.17\ \text{mm} = 3.917\ \text{cm} = 0.3917\ \text{dm} = 0.03917\ \text{m}$
5: The sequence included the replacing of quantities with specific values and calculating a value for the sequence	$u + v = (2,-1) + (-3,-2) = (-1,-3),$ $\Delta s_2 = v_m{\cdot}\Delta t = 4{\cdot}4\ \text{m} = 16\ \text{m}$

> We could also write the perimeter as $2(7a + 6b)$ because $7a + 6b$ is an expression for half the perimeter. That expression can be simplified in two ways. Laws that apply for numerical expressions also apply to algebraic expressions. $2(7a + 6b) = (7a + 6b) + (7a + 6b) = 14a + 12$. The parenthesis can be removed because there is a plus sign in front of it. (Gennow et al., 2017, p. 85, Authors' translation)

the first analysis unit, $2(7a + 6b)$, is categorised as a single mathematical expression (category 2) and not considered related to a preceding unit. The succeeding analysis unit, $7a + 6b$, is also categorised as a single mathematical expression (category 2) but considered related to the preceding unit. The next analysis unit, $2(7a + 6b) = (7a + 6b) + (7a + 6b) = 14a + 12b$, is also considered related to preceding unit, but this unit is categorised as a single mathematical relation (category 3, Table 2). In a similar way, when a mathematical relation is rewritten in the textbook, for example, 'The work is force times distance according to $W = F_s \cdot \Delta s$, the force then becomes $F_s = W/\Delta s$' (Fraenkel et al., 2011, p. 147, Authors' translation). The first analysis unit, $W = F_s \cdot \Delta s$, is categorised as a mathematical relation (category 3, Table 2) and it is not considered related to a preceding unit of analysis. Whereas the relation in succeeding analysis unit, $F_s = W/\Delta s$, also is categorised as a mathematical relation (category 3), but this analysis unit is also considered part of a preceding unit of analysis. In addition, as part of RQ2, the type of structural context in the textbook that included each symbolic sequence was noted, that is, if the sequence was a part of an example, a figure, a table, a definition box or the theory text.

To answer RQ3, words used in relation to a unit of analysis, either referring to the whole unit or parts of the unit (e.g. 'distributive law', 'velocity', 'force', 'multiplication with a scalar') were identified, as well as if the used words referred to physics or mathematics (when this was determinable). The distinction between physics words and mathematics words was based on in which school subject's respective syllabus (mathematics or physics at upper secondary school) the word, or related concept, is explicitly addressed. For example, 'angle' and 'volume' are covered in mathematics, whereas 'force' and 'distance' are addressed in physics.

Statistical methods were used to decide if there were any significant differences between the textbooks in mathematics and physics with respect to the different variables and categories created to answer the research questions. The two-sample t-test was used to compare means between the subjects and the Chi-square test of homogeneity was used to determine whether proportions of symbolic sequences were consistent between the subjects. In order to use a parametric test, such as the two-sample t-test, data should be normally distributed. From the Central Limit Theorem, the t-distribution tends to a normal distribution for large sample sizes, thus the normality condition could be neglected if the sample size is at least 30 (Sokal & Rohlf, 1987), which is the case for data in this study. Furthermore, for the chi-squared test for tables, the minimum expected number should be at least 5, but for two-by-two tables the 'N-1' chi-squared test can be used when the minimum expected number is at least 1 (Campbell, 2007). Analyses in this study are based on these conclusions and the data fulfil the given conditions. We use $p < .050$ as the criterion for statistical significance. In some parts of the analyses, we address a set of several tests of statistical significance. In such situations it is relevant to correct the criterion for statistical significance. We use the standard Bonferroni correction in such situations, that is, the criterion of 0.05 divided by the number of tests within the specific set.

Besides the statistical analysis, more exploratory analyses were also performed in relation to RQ2. Such analyses were used as the basis for the creation of the five categories mentioned above, but were also used to produce more descriptive results, to create a more comprehensive view of the use of algebraic symbols in mathematics and physics textbooks.

The categorisation procedure was developed and refined together by the two authors through analyses of some of the pages from the mathematics textbook and from the physics textbook. When consensus of the procedure was reached, the rest of the pages were divided between the authors and categorised separately. When all data were analysed, difficult situations were discussed and principles for how to categorise these were agreed on.

6. Results

In this section, we first describe the results from statistical analyses and details from more qualitative and in-depth characterisations, in relation to each of the three research questions. Finally, we give a short summary of the main findings from all analyses.

Overall, the symbolic sequences in the mathematics textbook varied from including one single algebraic object (e.g. x, $3a+7$) to one sequence with 18 algebraic objects, $a+2b+2a+3b+4a+b+7a+6b=a+2a+4a+7a+2b+3b+b+6b=14a+12b$. In the physics textbook, the symbolic sequences varied from including one single algebraic object (e.g. v, $I_1=12/6=2$ A) to one sequence with 23 algebraic objects, $F=(\rho g(h_1+h)\text{-}\rho g h_1)\cdot A=(\rho g h_1+\rho g h\text{ - }\rho g h_1)\cdot A=\rho g h A$.

6.1. Size and complexity of symbolic sequences (RQ1)

The results showed that there is on average a greater number of algebraic objects in a symbolic sequence in physics than in mathematics textbooks (2.4 vs. 1.9), and this difference is statistically significant (Table 3, row 1). A significant difference between mathematics and physics also occurred when comparing the average number of *different* algebraic objects in a symbolic sequence (Table 3, row 2). The analysis also showed that it is more common in physics textbooks that symbolic sequences that include more than one equal sign also include more different algebraic objects (3.4), than corresponding symbolic sequences in mathematics textbooks (1.7) (Table 3, row 4).

Table 3. Differences between mathematics and physics textbooks in the occurrence of algebraic objects and mathematical operations in symbolic sequences.

Aspect	Mathematics (*N*), mean, (*SD*)	Physics (*N*), mean, (*SD*)	*t* (*df*), *p*
Number of algebraic objects in symbolic sequences	(1014), 1.9, (1.6)	(296), 2.4, (2.7)	6.15 (1308), < .0001
Number of different algebraic objects in symbolic sequences	(1014), 1.3, (0.7)	(296), 2.1, (1.6)	11.90 (1308), < .0001
Number of algebraic objects in symbolic sequences including more than one equal sign	(65), 3.8, (4.0)	(60), 4.7, (4.5)	1.21 (123), .2272
Number of different algebraic objects in symbolic sequences including more than one equal sign	(65), 1.7, (1.2)	(60), 3.4, (2.0)	5.6 (123), < .0001
Number of mathematical operations in symbolic sequences that include any operation	(505), 2.9, (3.5)	(122), 4.0, (4.4)	2.85 (625), .0045
Number of different mathematical operations in symbolic sequences including at least two operations	(310), 2.2, (0.6)	(88), 2.0, (0.9)	2.71 (396), .0071

Table 4. Proportion of symbolic sequences in mathematics and physics textbooks with respect to inclusion of different types of algebraic symbols.

	Mathematics (N = 1014) %	Physics (N = 296) %	$\chi^2(1)$, p
Proportion of symbolic sequences including at least one index	6.3	50.7	329.78, < .0001
Proportion of symbolic sequences including at least one capital delta (Δ)	.0	18.6	196.51, < .0001
Proportion of symbolic sequences including at least one Greek letter (excl. Δ)	1.2	15.5	111.64, < .0001
Proportion of symbolic sequences including at least one equal sign	48.6	42.6	3.36, .0667
Proportion of symbolic sequences including more than one equal sign	6.2	19.9	51.04, < .0001

Furthermore, the analysis showed that there is on average more operations included in symbolic sequences in physics than in mathematics (4 vs. 3) (Table 3, row 5). However, when it comes to the number of different mathematical operations in a symbolic sequence, there is more in mathematics than in physics. Further analysis also showed that the types of included mathematical operations are similar in mathematics and in physics. Operations in symbolic sequences in the mathematics textbook were: addition, subtraction, multiplication, division, power functions (square, cube, n^{th}), root functions (square, cube, n^{th}) and trigonometric functions. Operations in symbolic sequences in the physics textbook were: addition, subtraction, multiplication, division, power functions (square), root functions (square), trigonometric functions, inverse proportionality and summation.

Moreover, as seen in Table 4, there is in general a higher degree of some special types of algebraic symbols in the symbolic sequences in the physics textbook compared to the mathematics textbook. More than half of all analysed physics units included the use of one or more indices, whereas indices were used in 6% of the analysed mathematics units (row 1); and in a fifth of all symbolic sequences in the physics textbook there was at least one Δ included, whereas this notion was not used at all in the symbolic sequences in the mathematics textbook (row 2). Other Greek letters were used in 16% of the symbolic sequences in physics compared to 1% of the symbolic sequences in mathematics (row 3). Furthermore, a fifth of the symbolic sequences in the physics textbook included more than one equal sign, whereas this property applied to 6% of the symbolic sequences in the mathematics textbook (row 5).

6.2. Characterisation of symbolic sequences (RQ2)

Comparing overall properties of symbolic sequences in mathematics and physics textbooks (Table 5), the analyses show that the overall distribution of symbolic sequences in the five categories are different between mathematics and physics; $\chi^2(4) = 158.44$, $p < .0001$. More specifically, post hoc analyses showed that there are significant differences between mathematics and physics regarding all the five categories. Furthermore, symbolic sequences that are part of a preceding unit of analysis are more common ($\chi^2(1) = 13.44$, $p = .0002$) in the mathematics textbook (35%) compared to the physics textbook (23%). These differences are explored in the following.

Table 5. Proportion of symbolic sequences in mathematics and physics textbooks with respect to overall properties (level of statistical significance is calculated based on the Bonferroni correction; $p < .050/5 = .010$).

Categories of overall properties	Mathematics ($N = 1012$) %	Physics ($N = 296$) %	$\chi^2(1), p$
1: Single algebraic object	34.2	50.0	24.34, < .0001
2: Single mathematical expression	12.6	3.0	22.21, < .0001
3: Single mathematical relation	36.9	23.3	18.78, < .0001
4: Specific value for an algebraic object	15.2	8.5	8.88, .0029
5: Calculate a value for the symbolic sequence with given values for algebraic objects	1.2	15.2	107.71, < .0001

6.2.1. Categories 1 and 2

Exploratory analysis of the data showed that about 70% of all analysed units in the mathematics textbook were symbolic sequences with one algebraic object. Among these symbolic sequences with one algebraic object, a majority (57%) did not include any equal sign (i.e. belonging to categories 1 and 2, Table 5) and almost a quarter (22%) of these sequences without any equal sign were constructed by the solitary alphabetical letter x. Overall, x appears as the most frequently used algebraic symbol in the mathematics textbook. Other commonly used letters to denote single algebraic objects in the mathematics textbook are v and y. Furthermore, mathematical expressions, such as $2x$ and $18 – 7x$, correspond to 15% of the symbolic sequences in mathematics that include one algebraic object, but no equal sign.

Similar analysis of the physics textbooks showed that a majority (appr. 60%) of the physics units consisted of one algebraic object, and, similarly as for the symbolic sequences in the mathematics textbook, a majority (83%) of these units with one algebraic object did not include any equal sign (i.e. belonging to categories 1 and 2, Table 5). Exploratory analysis showed that unlike symbolic sequences with a single algebraic object in mathematics, there are no typical alphabetical letters or other typical algebraic symbols used in these symbolic sequences in physics. Instead, used algebraic symbols seem to relate to what page that is analysed. For example, on page 89 (Fraenkel et al., 2011) only F is used as a single algebraic symbol, whereas on page 269 (Fraenkel et al., 2011) algebraic symbols for single algebraic objects are ΔE_{in}, ΔE_{out}, Δh and ϵ. However, similar to the mathematics textbook, half of the symbolic sequences without any equal sign and just one algebraic object consist of a single letter (e.g. F, U, v) in the physics textbook, and on a few occasions in combination with a Δ (e.g. Δs); and almost 90% of the other half of the symbolic sequences without any equal sign and just one algebraic object consist of a single letter with an index, such as F_1, F_{down}, R_y and sometimes in combination with Δ (e.g. ΔE_{in} and Δs_2). As described above, unlike symbolic sequences with a single algebraic object in mathematics, there were no mathematical expressions among symbolic sequences in the physics textbook that consisted of a single algebraic object.

The proportion of symbolic sequences as single mathematical expressions was larger in mathematics than in physics (13% vs. 3%, category 2). Deeper analysis of the data showed that typical examples of mathematical expressions in the mathematics textbook are $2x + 11$ and $3u$-$2v$, and that practically all expressions either included one or two algebraic objects. A typical example of a mathematical expression in the physics textbook is $m_1 g$, and in physics, all mathematical expressions consisted of two different algebraic objects.

6.2.2. Category 3

As shown in Table 5, the proportion of single mathematical relations was higher in the mathematics textbook (category 3). Exploratory analysis showed that there is a difference between mathematics and physics regarding in what context in the textbooks these single mathematical relations occur. In the mathematics textbook, about 65% of the single mathematical relations are in examples of how to solve various types of equations. In total, a fifth (19%) of the symbolic sequences in category 3 come from the main text in the mathematics textbook, and in this context, symbolic sequences mainly consist of functions (e.g. $f(x) = 20x$), formulas (e.g. $A = \pi r^2$), rules (e.g. $a(b + c) = a \cdot b + a \cdot c$) and manipulations of an expression (e.g. $2b + 3b + b = 6b$). Symbolic sequences of the form functions, formulas and equations occur in the highlighted 'definition boxes', where 10% of the total number of entities in this category came from these 'boxes'. In the physics textbook, on the other hand, more than half (58%) of the single mathematical relations occur in the main text and describe a relation between various quantities (e.g. $U = R \cdot I$ and $E_p = mgh$), and almost a fourth of the single mathematical relations appear in 'definition boxes' in the same form as in the main text (e.g. $F_{\text{lift}} = \rho gV$). Similar relations are also used in examples in the textbooks, corresponding to 16% of the symbolic sequences in the physics textbook categorised as single mathematical relations.

6.2.3. Category 4

Of the symbolic sequences in the mathematics textbook that include one algebraic object, and *at least one* equal sign, 90% consisted of one equal sign, and of these, almost half were of the type $x = 34.5$ (category 4, Table 5). In the physics textbook, almost half of the symbolic sequences that include one algebraic object, and *at least one* equal sign included one equal sign (e.g. $m = 47.952533$ u, $\Delta h = 300$ m and $v_{A2} = -2$ m/s), and the other half included two equal signs (e.g. $b = 72$ mm $= 0.072$ m and $I_1 = 12/6 = 2$ A), both belonging to category 4 (Table 5). As shown in Table 5, this category is more common in the mathematics textbook. A deeper analysis shows that in mathematics, almost none of the values is associated to a unit, whereas this applies to almost all of the values in physics.

6.2.4. Category 5

As shown in Table 5, there is a larger proportion of symbolic sequences that involves replacing algebraic objects with given values and/or deriving a value for the sequences in the physics textbook (15%) than in the mathematics textbook (1.2%). Deeper analysis showed that in the physics textbook, these symbolic sequences are typically $\Delta s_3 = v_{\text{m}} \cdot \Delta t = -2 \cdot 3\,\text{m} = -6\,\text{m}$, and $A_{\text{elephant}} = 4 \cdot \pi r^2 = 4 \cdot \pi \cdot 0.25^2\ \text{m}^2 = 0.7854\ \text{m}^2$, and most of the analysed units (80%) come from examples of how to solve various tasks. Although there is a smaller proportion of this type of symbolic sequences in the mathematics textbook, just as in the physics textbook, these sequences are found in examples of solutions to specific tasks, for example $m = 80{\cdot}0.5^{(8000/1600)} = 80{\cdot}0.5^5 = 2.5$. In the same way as above (category 4), almost none of the derived values in mathematics is associated with a unit, whereas this applies to all derived values in physics.

6.2.5. Symbolic sequences that are part of a preceding unit of analysis

Regarding symbolic sequences that are parts of a preceding unit of analysis there is a larger proportion in the mathematics textbook (35%), compared to the physics textbook (23%). Further exploratory analysis showed that in the mathematics textbook, 87% of this kind of symbolic sequences were found in examples of how to solve various types of equations, typically, '$x/9 = 62$' 'Multiply both sides with 9', and 'Answer: $x = 558$', where $x = 558$ is considered to be part of preceding unit of analysis. In the physics textbook, unlike in the mathematics textbook, 44% of the symbolic sequences that are part of a preceding unit of analysis came from the main text, and mostly when previously introduced algebraic objects were further explained or relations were derived (e.g. $W = mg(h_2\text{-}h_1) = mgh_2\text{-}mgh_1 = E_{p2}\text{-}E_{p1}$). Another major part (39%) of the symbolic sequences that are part of preceding unit of analysis in physics, was found in the 'definition boxes' as explanations of already introduced algebraic objects. For example, 'ΔE_{out} is the useful heat energy'. The remaining symbolic sequences in the physics textbook noted to be part of preceding unit of analysis, were found in examples (17%) and consisted of manipulations of relations and calculations with specific values.

6.3. Types of words that are used in relation to symbolic sequences (RQ3)

Results showed that there are clear differences between mathematics and physics concerning if and how words are used to label or name the symbolic sequences, or parts of the sequences (Table 6). In mathematics, more than half of the symbolic sequences are never referred to by using any words, while the same is true for 18% of the sequences in physics. Overall, mathematical words are usually used in the mathematics textbook and physics words are usually used in the physics textbook.

Deeper analysis showed that the physics words that exist in the mathematics textbook mostly refer to time (15 instances) and distance (10 instances), while singular instances refer to mass, resistance, voltage and current. The physics words in the physics textbook mostly refer to force (38 instances) and energy (29 instances). Distance is also referred to several times (18 instances), while time is only referred to in 4 instances. That is, time is the most common physics word used in mathematics, but it is not that common in physics, when addressing symbolic sequences.

Furthermore, analyses also showed that regarding mathematical words that exist in the physics textbook in relation to symbolic sequences, words from geometry are most common; volume (9 instances), area (3), angle (3), height (3), diameter (1) and circumference (1). There are also words from algebra; formula (3) and equation (1), and also words concerning vectors; components (5) and resultant (3). The mathematics words in the mathematics textbook mostly refer to algebra, with equation (66 instances), expression

Table 6. Proportion of symbolic sequences in mathematics and physics textbooks with respect to type of words used as referent.

	Mathematics ($N = 1014$) %	Physics ($N = 296$) %	$\chi^2(1), p$
No used words	53.7	18.2	115.70, $< .0001$
Mathematical words	42.3	11.5	95.16, $< .0001$
Physics words	3.5	74.7	738.42, $< .0001$

(35), variable (32) and function (27) as the most common. There are also words connected to vectors (in total 66 instances), geometry (33 instances) and a variety of other names of different types of numbers or expressions (in total 66 instances), such as fraction, decimal, exponent, rote, power or product. A majority of words for symbolic sequences in mathematics related to algebra, which is likely a natural consequence of our selection method, with our focus on the handling of algebraic symbols in certain types of contexts (see method description above).

6.4. Summary of results (all RQs)

The analyses of the size and complexity of symbolic sequences show many differences between mathematics and physics (RQ1). Overall, the results show that symbolic sequences tend to be longer and more complex in physics, compared to mathematics. For example, symbolic sequences in physics tend to have a higher number of algebraic objects in total, of different algebraic objects, of mathematical operations in total, of different mathematical operations and of equal signs.

The characterisation of symbolic sequences shows many differences between mathematics and physics concerning what types of sequences are used (RQ2). It is more common in physics to present single algebraic objects while it is more common in mathematics to present single mathematical expressions or relations. When focusing on specific values of algebraic objects, it is more common in physics to calculate a value for a symbolic sequence with given values for several algebraic objects, while it is more common in mathematics to present a specific value for a single algebraic object.

The analyses of the types of words that are used to label or name symbolic sequences, or parts of sequences, show many differences between mathematics and physics (RQ3). It is much more common in mathematics to not use any words at all in relation to symbolic sequences, which happens for more than half of the sequences in mathematics, but only for 18% of the sequences in physics. When words are used, mathematical words are usually used in the mathematics textbook and physics words are usually used in the physics textbook. Furthermore, the types of words used in physics usually refer to a physical magnitude, such as time, force or energy, while the words used in mathematics usually refer to a name of the type of symbolic sequence, such as equation, expression or function.

7. Discussion

Here we discuss our results from different perspectives. First, we relate our empirical analyses to different statements in research literature. Second, we use our results to discuss if and how students meet different algebra discourses in mathematics and physics. Finally, we discuss issues around doing comparisons between mathematics and physics.

7.1. Validity of statements in research literature

As described in the background, there exists different statements in research literature about differences in algebra between mathematics and physics. Such statements have not been backed up by structured empirical analyses, which we now have produced. So, do our empirical analyses support the statements in previous research literature? Overall, the answer is yes, which we describe from three perspectives, connecting to the three research questions.

First, in relation to RQ1, there are statements in previous research that mathematics usually focuses on single algebraic symbols while there are usually many different algebraic symbols in physics (Heck & van Buuren, 2019; Redish, 2006). This type of difference is supported by our empirical analyses that show significant differences between mathematics and physics concerning the number of different algebraic objects in sequences, both in general and also for more complex sequences (when several equal signs are used). Also, our analyses show that it is more common in physics than in mathematics to use special types of symbols (index, constructions with delta and Greek letters).

Second, in relation to RQ2, there are general statements in previous research that the use of symbols is different in mathematics and physics (Redish & Kuo, 2015). Our analyses support these statements, concerning the more overall characterisation of the symbolic sequences. In physics, single algebraic objects are more common and also situations where a value is calculated for a symbolic sequence with given values for algebraic objects. On the other hand, the mathematics textbook to a larger extent uses single mathematical expressions and single mathematical relations. It is also more common in mathematics with situations that specify values for an algebraic object and situations where a symbolic sequence is connected to a previously given symbolic sequence. On the contrary, physics tends to build larger or more complex symbolic sequences, since it is more common in physics with sequences that include more than one equal sign. A similarity between mathematics and physics regarding the use of algebraic symbols concerns symbols for operations. Although there are slightly more symbols for operations included in the symbolic sequences in physics, there is large similarity between the subjects concerning the types of operations that are used in mathematics and physics. This can connect to previous research where it is noted that the same formal syntax for symbolic sequences is used in both subjects and students do not seem to have greatest difficulty with the manipulation of symbols, but the main difficulty could be about the meaning of symbols (Redish & Kuo, 2015; Torigoe & Gladding, 2011). These results of differences in how symbols are used lead to a need for more in-depth analyses to examine if there are other structures of similarities and differences. For example, the different roles of the equal sign (cf. Knuth et al., 2006) might be a relevant starting point. Each of our categories can include different uses of the equal sign, for example, our category 3 involves the use of the equal sign as part of an equation or as part of a formula. An analysis of the meaning of the equal sign would then create another set of categories, which might reveal important similarities and differences between the use of symbols in mathematics and physics.

Third, in relation to RQ3, there are statements in previous research that the loading of meaning to symbols is more common in physics (Ellermeijer & Heck, 2002; Redish & Kuo, 2015). Our analyses of the words used when addressing algebraic symbols support these statements. In mathematics, there is often no word used at all to address a symbolic sequence, while words are used in this way for most sequences in physics. Also, the words used in physics mostly concern a type of referent for a symbol, such as energy or force, while the words used in mathematics mostly are general names for the (type of) symbol or symbolic sequence used, such as equation or variable. One way to characterise this difference, as we see it, is that mathematics mostly *talks about the symbols*, while physics mostly *talks through the symbols*, concerning their meaning.

7.2. Students' meetings with algebra in mathematics and physics

Based on our comparative analyses of textbooks, can we say that students meet different algebra discourses in mathematics and physics? We relate to Sfard's (2008) characterisation of what can distinguish different discourses to address this question. Our analyses have focused on two of Sfard's characteristics: visual mediators and word-use. We have focused on a certain type of visual mediator, concerning algebraic symbols, primarily symbolically represented indeterminate quantities (cf. Radford, 2018). Our analyses show that there are differences between mathematics and physics concerning these types of visual mediators, where the symbolic sequences are more varied and more complex in physics and where the symbolic sequences are used in different ways in the different subjects. For word-use, our analyses have shown that there are differences between mathematics and physics concerning how words are used to address algebraic symbols. It is uncommon in mathematics, but common in physics, to use words to address algebraic symbols. When using such words, physics tends to use words that focus on the meaning of symbols, while mathematics primarily focuses on the general naming of different types of symbols or symbolic sequences. That is, we see differences in the algebra discourse between mathematics and physics, concerning the types of visual mediators, how these visual mediators are used, and also how words are used to address these visual mediators.

However, there are also similarities between the subjects. For example, our result shows that the same variation and types of operations are used in mathematics and physics. Also, the type of symbols and sequences are similar, since they all are or include symbolically represented indeterminate quantities. Therefore, the same type of rules for manipulation are valid in both subjects. That is, the basic syntax and grammar for symbols are the same. This similarity concerns a more abstract property of the *type* of symbols, while our analyses have shown differences concerning the specific symbols that are used.

In summary, there is an overlap between the algebra discourses in mathematics and physics, but they are not the same and the differences are at a high level of specialisation. Therefore, we support the conclusion drawn by other researchers (Karam et al., 2019; Redish & Kuo, 2015) that it is not fruitful to think only of transfer from mathematics to physics concerning students' experiences and knowledge of algebra. Our analyses have shown several differences between the subjects that students need to handle when using their textbooks. Since these differences concern core aspects of the algebra discourse, it can be difficult for students to see the similarities between algebra in mathematics and algebra in physics.

Using the specific results from our analyses as a starting point, it would be valuable to examine how students use the textbooks in the different subjects, especially how words are used by the students to address the symbols. It would also be valuable to include analyses of teachers, especially teachers that teach both subjects, to examine if and how their algebra discourses are different in mathematics and physics. A modelling situation could be of particular relevance to examine, since modelling can include phases where focus is on the meaning of symbols, which we have seen mostly in the physics textbook, and phases where focus is on the manipulation of symbols where a more general characterisation of symbols might be more evident, which we have seen mostly in the mathematics textbook. Therefore, it would be relevant to examine if and how the algebra discourses among students and teachers change depending on the modelling phase and how the transition between discourses can be characterised, for example, as a continuous or a more distinct transition. Results from this type of more in-depth empirical analyses could then also be related to other studies that have addressed more overarching issues of modelling as a perspective on relations between mathematics and physics (Hansson et al., 2015; Redish, 2006; Uhden et al., 2012).

7.3. Comparisons between mathematics and physics

Our analyses have shown many differences between the subjects regarding algebraic discourse. These differences are in a way not surprising, since we analyse different subjects that may have different characteristics, traditions and needs. For example, in mathematics education, a problem has been discussed when using letters as representing physical objects, functioning as names of objects, rather than values (e.g. Arcavi et al., 2017). This has been identified to be a result of 'fruit salad algebra' (Arcavi et al., 2017, p. 51), where teachers chose letters as abbreviations for objects, such as variable b when handling the number of bananas. In physics, however, letters are deliberately used to relate more directly to the meaning of the symbol, such as E for the amount of energy, which is considered an advantage (cf. Ellermeijer & Heck, 2002; Redish & Kuo, 2015).

However, if you consider mathematics as a service subject, it becomes more unclear why there are not more similarities in the algebraic discourses. For example, a relevant similarity could be about handling somewhat more complex algebraic symbolic expressions in mathematics teaching. At the same time, there can be complexity in acting as a service subject to potentially many different subjects, with variations in their traditions and needs.

A potential direct implication of our results for textbook authors and teachers of mathematics could be to also include more complex algebraic symbolic expressions in textbooks and in teaching. For example, symbolic expressions could be used that are longer and consist of more algebraic objects, as these are more common in physics. This type of symbolic expressions are needed in physics, among other things in the formulas and relationships that describe the physical phenomena included in the subject. Furthermore, our results reveal differences and similarities between the subjects, as described above, which teachers need to be aware of in order to explicitly address these to the students and thus improve teaching and learning (Heck & van Buuren, 2019).

We have analysed parts of some Swedish textbooks to compare the algebra discourse in mathematics and physics. A central question is how representative our results are for broader perspectives, such as other parts of textbooks, textbooks by other authors, other parts of teaching and other countries. From a technical perspective, our statistical analysis only allows conclusions about the specific textbooks that we have analysed. However, certain things point to a broader relevance of our analyses and results. The textbook has a central role in teaching, both in mathematics (Fan et al., 2013) and in science (Vojíř & Rusek, 2019). Thus, analysis of textbooks can produce a relevant picture of the discourse in the subjects. Furthermore, comparative analyses of textbooks have also shown great similarities between different countries, at least regarding types of tasks in mathematics

(Jäder et al., 2020). In addition, our analyses have proven to be in line with more anecdotal descriptions from researchers from different countries.

8. Concluding remarks

We have contributed with a structured empirical comparison between mathematics and physics, focusing on the use of algebraic symbols in textbooks. This type of analysis has been missing in research, which previously has mostly focused on physics, and the role of mathematics in physics, and as discussed above, our results have direct implications for teaching and learning. The discourse perspective we have adopted in this study (Sfard, 2008) has several benefits in contributing to the research field concerning relationships between mathematics and physics. First, it gives a tool for focusing the analyses on specific properties of the subjects, in our case certain types of visual mediators (algebraic symbols) and word-use in relation to these visual mediators. Second, there is no need to address 'mathematics' in more general terms, which often become too abstract, when examining relationships between mathematics and physics. Third, this perspective also allows for analyses of variations *within* subjects, for example, to examine if and how there are different algebra discourses in different parts of mathematics. Such analyses could also reveal if there are certain parts of mathematics that are more similar to (certain parts of) physics, concerning the use of algebraic symbols. Thus, we see the need for more structured empirical and comparative analyses of mathematics and physics in school, to understand more about the relationships between different discourses, which can allow for a deeper understanding of how students handle such discourses.

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