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From process to object in teachers' introductory algebra discourse

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ABSTRACT

Generalization is one of the core aspects of algebraic thinking, and research literature points to the importance of objectification when generalizing mathematical discourses. To increase the field's understanding of the objectification process in various algebra discourses, our study analyses Grade 6 teachers' use of words, symbols, routines and narratives when they introduce various algebraic concepts. Our results show that there exist instances of objectification in the teachers' discourses, but these are all implicit, which can make them difficult for students to notice. Thus, teachers need to be made more aware of the importance of making the objectification process explicit. Furthermore, our results show that instances of objectification are tightly connected to the use of formal algebraic symbols. This highlights the importance of the transition from concrete/iconic to formal/arbitrary representations.

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KEYWORDS
Mathematical discourse; mathematical symbolization; objectification

1. Introduction

Algebra is one of the most difficult mathematical areas for many students, including Swedish students (e.g. Olsen & Grønmo, 2006; Swedish National Agency for Education, 2013, 2016), and an early introduction to algebra has been shown to be a gatekeeper for success in subsequent mathematics courses (e.g. Lee & Mao, 2020). An international movement over the last 20 years has been to integrate algebraic reasoning into the elementary school curriculum prior to the introduction to formal symbolism (Kieran, 2018). In the Swedish school system, some algebraic thinking based on the use of informal representations is nowadays introduced in early grades, while students meet more formal symbolism at the end of middle school (age 12). The present study investigates precisely the transition from informal to formal symbolism in Swedish algebra classrooms.

The traditional introduction to formal algebra typically involves becoming acquainted with variables represented by alphanumeric symbols, algebraic expressions and equations, as well as methods for solving algebraic equations. This resembles Mason's (2011) description of traditional algebra 'as arithmetic with letters, dominated by procedures for manipulating symbols' (p. 561). It is, however, not always sufficient if students are to learn

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algebra in a profound and generalizable way. In a large international comparative research project (Kilhamn & Säljö, 2019), introductory algebra lessons, in the phase of moving from informal to formal algebra, were video recorded with the aim of seeking hidden dimensions of algebra teaching and learning. When analyzing teaching approaches and lesson structure, large variations were found concerning, for example, the amount of written work students were exposed to. Among many interesting observations, we were struck by the different ways in which algebraic symbols were introduced and explored in the Swedish Grade 6 classrooms, and therefore set out to explore these differences further.

It is widely agreed that teachers play a central role for students' learning and their development of a mathematical discourse (e.g. Blanton et al., 2018; Nachlieli & Tabach, 2022; Sfard, 2016; Tabach & Nachlieli, 2016). In an investigation of prospective teachers during a course about early algebra, Hohensee (2017) showed that while students need to transition from arithmetic to early algebra, and eventually to formal algebra, the prospective teachers found it challenging to move from algebraic symbolism to informal diagrams and pictures. Much has also been said about the importance of enhancing language to promote mathematical learning (Erath et al., 2021). Thus, how teachers introduce various algebraic concepts and symbols, and what type of algebraic discourse they model, is of interest in order to understand students' algebra learning. This paper contributes by taking a discursive perspective on middle-school teachers' mathematical instruction. By analyzing differences in the objectification process in the discourse developed by teachers during introductory algebra lessons in Grade 6, this study aims to discuss potential opportunities or obstacles for algebra learning. Before going into the empirical part of the article, we will highlight what previous research has shown concerning students' use of algebraic symbols and introduce a discursive perspective on algebra learning.

2. Background

A core aspect of algebraic thinking is generalization (Blanton et al., 2018; Caspi & Sfard, 2012; Kaput et al., 2008; Mason et al., 1985). According to Radford's (2018) definition, algebraic thinking is about dealing with indeterminate quantities in an analytical way, which implies that, despite being unknown, the quantities are operated on as if they were known. Indeterminate quantities are used to denote that more than given numbers or other mathematical entities can be involved in algebraic situations. These 'quantities can be unknowns, variables, parameters, generalized numbers, etc'. (Radford, 2018, p. 8).

There are different views on the role of symbolization in algebraic thinking. For some researchers, alphanumeric symbols are required for an activity to be considered genuinely algebraic (Blanton et al., 2017; Kaput et al., 2008), while for others they are not (Mason et al., 1985; Radford, 2014). Caspi and Sfard (2012) distinguish between informal and formal algebraic discourse, where symbolization is one major visible difference between the two forms. Not requiring alphanumeric symbols allows for a broader definition of what counts as an algebraic symbol, including, for example, informal symbols, natural language, gestures, rhythm and other non-traditional semiotic systems (Radford, 2018). While each semiotic system constrains what can be expressed in the particular system, it is agreed that the alphanumeric symbol system is very powerful, and therefore important in helping students experience structure. When algebraic symbols are used in mathematics, they function as referees that emphasize general characteristics and exclude extraneous attributes (Mason, 2018), and convey different meanings in different situations (Ely & Adams, 2012). According to Caspi and Sfard (2012), the alphanumeric symbol system enables and amplifies objectification.

2.1. Students' use of algebraic symbols

Students' understanding of alphanumeric symbols in relation to their learning of algebra is an intriguing field of study. Interviewing first-grade students, Blanton et al. (2017) identified six levels of sophistication describing students' understanding of letters as representing variables. The different levels span from no experience of indeterminate quantities to the ability to 'act on variables, represented with letters, as mathematical objects' (Blanton et al., 2017, p. 196). One level described the use of letters as representing physical objects, functioning as names of objects, rather than values. This was also noticed by Caspi and Sfard (2012). It has been identified to be a result of 'fruit salad algebra' (Arcavi et al., 2017, p. 51), where teachers chose letters as abbreviations for objects, such as variable *b* when handling the number of bananas. Another intermediate level of sophistication described by Blanton et al. (2017) includes the perception that letters represent fixed values, and that the represented number is connected to the letter, for example the letter's position in the alphabet. This was also identified by Rystedt, Kilhamn and Helenius (2016) when analyzing students' discussing how to write an expression for a number of items expressed in *n*. One student's initial strategy was to count which number the letter *n* had in the alphabet. Blanton et al. (2017) concluded that in order to develop the ability to act on variables as mathematical objects, students need experiences of mathematical situations involving a variable quantity, where symbolizing the variable is a necessity. Furthermore, students' development of algebraic thinking could be said to follow a process-to-object trajectory (Blanton et al., 2017). Many studies have also described the importance of moving from a processual/operational meaning of the equal sign to a relational meaning (e.g. Prediger, 2010), which entails a transition from talk about the process of 'making equal' to talk about 'equivalence' as an object.

A similar description of the development of students' algebraic discourse from processual to objectified is provided by Caspi and Sfard (2012), in a model consisting of hierarchical layers, empirically developed by analyzing Grade 5 and Grade 7 students' discussions when solving various tasks concerning constant value algebra (e.g. generalizing patterns). Furthermore, they argue that 'an introduction of a new layer before the student mastered the preceding one carries the risk that the learner would simply not know what the new discourse is all about' (Caspi & Sfard, 2012, p. 47).

Steinweg et al. (2018) highlight challenges in designing algebra tasks. They analyzed students' responses to tasks of different levels of difficulty, with a focus on students' relational thinking (e.g. the ability to describe relations between known and unknown quantities). In the tasks, known quantities were represented by a visible number of marbles and unknown quantities appeared as boxes containing an arbitrary number of marbles. In some tasks, it was only possible to describe these numbers as a relation, in contrast to other tasks where variables appeared as determinable unknowns. By analyzing students' performance, Steinweg et al. (2018) concluded that tasks that leave the numerical value ambiguous supported students' relational thinking, and thus their algebraic understanding. Furthermore, they suggest that operating with marbles and boxes as concrete representations of numbers and variables can help visualize different approaches to a solution. However, in a small case study, Rystedt, Helenius and Kilhamn (2016) found that students who adopted a similar approach, using boxes and beans when solving an equation, failed to properly interpret the numerical solution they found in terms of the original context.

From the above-mentioned studies, we see that the role of variables, as well as the way variables are symbolized, are important features in students' learning of algebra. The results also imply a somewhat hierarchical development of the use of algebraic symbols, from informal to formal representations.

2.2. *Objectification and changing mathematical discourses*

The importance of perceiving mathematical concepts as both processes and objects has been addressed from various perspectives over the years (e.g. Dubinsky, 1991; Gray & Tall, 1994; Sfard, 1991, 2008, 2020). Much of this research has focused on individuals' learning from a cognitive perspective without substantial consideration of context. This limits the understanding of students' learning provided by these types of studies since, as Sriraman and Nardi (2013) point out, there is no context-free learning. At the same time, the individual perspective on thinking and learning is not to be forgotten. By taking a discursive perspective that treats thinking as a special type of interpersonal communication, tenets from both individual and social perspectives are merged (Sfard, 2008, 2020, see theoretical framework below). Learning, from this perspective, is considered to occur when an individual experiences discursive changes, for example when mathematical processes are discursively turned into objects. This is called objectification of the mathematical discourse.

When studying students' learning of functions, Nachlieli and Tabach (2012) showed that objectification is a complex process. Students followed a two-month introductory algebra course designed on the assumption that 'learners would have to engage in the conversation about function as a precondition for objectification of the focal signifier function' (Nachlieli & Tabach, 2012, p. 15). Although exposed to the word 'function' through teachers' utterances and task formulations, students rarely used the word themselves. Instead, they could keep the conversation going by remaining in the already objectified discourses of graphs and symbolic expressions. The students were mainly action-oriented and focused on what to do rather than the meaning of function on a metalevel. Nachlieli and Tabach (2012) concluded that, in order to promote objectification, teachers must introduce a new discourse gradually and consistently, and allow students to reflect on the discursive elements.

From a discursive perspective, the teacher models the classroom discourse, expecting students to follow and change their discourse accordingly (Nachlieli & Tabach, 2012, 2022; Tabach & Nachlieli, 2016). Sfard (2016) found that teachers can support students' learning (i.e. foster discursive changes) explicitly, or model the discourse implicitly. Explicit support, with a focus on the relation between teaching and learning, has been studied, for example by Güçler (2016), who showed that teaching situations in which mathematical discourse was explicitly addressed on a metalevel supported students' learning. Through becoming aware of particular metalevel rules, the students could identify the nature of the difficulties they struggled with. Shinno (2018) showed that teachers can also explicitly support students' objectification process by consciously modifying questions. In the study, the discourse was gradually objectified by the students when a computational process was discursively turned into an irrational number.

By comparing teachers' and students' discourses, results of implicit modelling have been studied. For example, Emre-Akdoğan et al. (2018) showed that the teacher's lack of explication of metalevel aspects of the discourse hindered students in objectifying their discourse on geometric translation. The students adopted some instances of the teacher's discourse, but abandoned elements in the discourse that were not compatible with their existing realization of the mathematical concept. Similar results were found by Güçler (2013, 2014), concerning the distinction between process and object aspects of the limit concept. A contrasting result, in which students' and their teacher's discourses showed great resemblance, is described by Sfard (2016). In that case, the teacher's discourse did not show any instances of objectification and one can assume that much of the teacher's discourse already matched the students' discourse, thus no discursive changes were required.

Other studies have focused on properties of teachers' mathematical discourse in various teaching situations. For example, Güçler (2014) analyzed one university teacher's discourse on the limit notation and found that the discourse around particular parts of the notation was processual, for example when exploring the behaviour of functions near the limit points, while that around others was objectified, for example when completing calculations of the limit-value. Johansson and Österholm (2019) found that upper secondary teachers' verbal discourse around symbols was mostly objectified when symbols that could be regarded as more familiar to the students were used, for example talking about the equal sign and about algebraic symbols in equation, but when no symbols had been introduced for familiar concepts, such as a formal sign for equivalence, the discourse was mostly processual, for example in the process of handling expressions or solving equations.

3. Theoretical framework

This study draws on Sfard's (2008) commognition framework, where 'thinking is defined as the individualized version of (interpersonal) communication' (p. 81). Within this perspective, thinking and speaking are inseparable. This implies that discourse, developed through interaction as a means of communication, is also seen as individual. Some patterns in an individual's public discursive behaviour (i.e. how an individual communicates with others) are assumed to remain stable although interlocutors change. Changes in the individual's mathematical discourse are seen as an indication of learning. This implies changes in 'ways of doing things' (Lavie et al., 2019, p. 159) and that 'learners gradually become capable of employing the discourse agentively, in response to their own needs' (Sfard, 2020, p. 2). Obviously, there are many other approaches to learning, but in this article, Sfard's (2008, 2020) commognitive perspective is taken as a point of departure.

One goal of mathematics education is thus to 'change elements in students' discourses so that they can participate in the historically established activity of mathematics' (Güçler, 2013, p. 440). From a teacher perspective, the 'goal is for the students to become expert participants in the mathematics classroom discourse' (Nachlieli & Tabach, 2022). For a discourse to be considered mathematical, communication should be about mathematical objects, such as *three*, *limit value*, *variable*, *equation*, *function* etc. (Sfard, 2008, 2020). In this study, the focus is on algebraic discourse, defined as a discourse dealing with symbolically represented indeterminate quantities and relations between them (Radford, 2018).

There are four characteristics that identify a particular discourse: *word-use, visual mediators, endorsed narratives* and *routines* (Sfard, 2008, 2020). Word-use refers to mathematical vocabulary unique to a particular mathematical discourse, such as 'variable' or 'algorithm', but also to colloquial words that are used with particular meanings, for example 'secret number' to explain variable or 'become' as a processual interpretation of the equal sign (Sfard, 2008, 2020; Tabach & Nachlieli, 2016). Visual mediators are the visible objects that are developed for mathematical communication, for example diagrams and symbols (Sfard, 2020; Tabach & Nachlieli, 2016). Endorsed narratives include statements describing objects, and relations, which are agreed to be true in the particular discourse, such as theorems and proofs, as well as computational rules (Sfard, 2020). Routines are metalevel rules that describe patterns typical of the particular discourse, for example how to solve equations or calculate the value of an algebraic expression (Sfard, 2020; Tabach & Nachlieli, 2016).

From a commognitive perspective, *objectification* is regarded as a 'process in which a noun begins to be used as if it signified an extradiscursive, self-sustained entity (object), independent of human agency' (Sfard, 2008, p. 300). There are two sub-processes that constitute the process of objectification: *reification* – 'the act of replacing sentences about processes and actions with propositions about states and objects' (p. 44), and *alienation* – the process when 'the alleged products of the mind's actions may undergo the final objectification by being fully dissociated, or alienated, from the actor' (p. 50). As an example of reification, compare the sentence 'In Newton's theory, the word 'force' *was used differently* than in the Aristotelian physics' (p. 44), with a reified version 'The word 'force' *had a different meaning* in the Newtonian and Aristotelian theories' (p. 44). As an example of alienation, compare the sentence '*We shall call* a polygon a triangle if and only if it has three sides' (p. 57) with an alienated version 'A polygon *is* a triangle if and only if it has three sides' (p. 57). Sfard (2008) stresses that although reification and alienation are two tightly related processes, 'these two types of transformation are attained by different discursive means, and the occurrence of one of them does not necessitate the other' (p. 44). A discourse is characterized as being *process-oriented* when it focuses on processes and actions, and *object-oriented* when it focuses on states and objects (Sfard, 2008).

Learning can occur on the object level, where the existing discourse about mathematical objects is expanded without questioning the already existing aspects, or on a metalevel, where the visual mediators, narratives and rules for word-use change, and new discursive patterns evolve (Sfard, 2008, 2020). An example of learning at the object level is 'realizing that the expression $2(x+y)$ could also be written as $2x+2y$' (Tabach & Nachlieli, 2016, p. 302). Learning at the metalevel on the other hand, could be to start using a mathematical definition of a rectangle instead of basing identification on visual similarities with other rectangles (Tabach & Nachlieli, 2016).

4. Purpose and research questions

As shown in the background, objectification is a complex process, and teachers' discursive treatment of mathematical concepts depends on the use of symbols. Furthermore, teachers explicitly addressing metalevel aspects of the discourse seems to support students' objectification, while implicitly modelling the discourse teachers want their students to adopt may provide limited opportunities for learning. A fundamental assumption of this study is that a teacher's discourse has an impact on the discourse that emerges in the classroom, although there is still more to learn about the nature of that impact (Sfard, 2016). By analyzing algebra discourses of three different teachers, we attempt to identify discursive differences, which in turn could have an impact on students' opportunities to develop an objectified algebraic discourse. The purpose of this study is to explore potential opportunities or obstacles for algebra learning from a discursive perspective. We look in detail at the teachers' oral and written discourse around variables, expressions, equations and algebraic symbols, highlighting processes of reification and alienation. Specifically, we address the research question (RQ): *What characterizes a more or less objectified algebraic discourse in the introduction of school algebra in middle school?* The results are discussed in terms of students' opportunities to develop an objectified algebra discourse.

5. Method

This is a case study investigating the discourse of three teachers as it appeared in their introductory algebra instruction in Grade 6 (students' age: 12 years). Data originates from an international research project about algebra teaching in the phase when moving from informal to formal algebra (Kilhamn & Säljö, 2019). From the Swedish data, three Grade 6 teachers were picked out because their teaching appeared to be quite different although they followed the same curriculum and used the same textbook. Data consists of video recordings of the first four lessons taught on the topic of algebra in the autumn of 2011. Based on observations and lesson plans, a graphic overview of the flow and content of each lesson was constructed (see Appendix A–C for examples). To capture the teacher's discourse, a video camera followed the teacher during the entire lesson, with a microphone attached to the teacher, and detailed transcripts were made of every utterance the teacher made.

5.1. Participants

The participating teachers worked in three different schools in and around a big city in Sweden, each teaching mathematics along with other subjects in Grade 6. Informed consent was given by the teachers, the students and their parents. Details about the teachers and their lessons are summarized in Table 1.

5.2. Process of analysis

The transcripts were analyzed in parallel with the actual video so that teachers' gestures and written work were included. Two researchers, first individually and then together, analyzed the teachers' discourse to identify word-use, visual mediators, routines and narratives related to algebra. Once these had been identified, differences and similarities concerning objectification in the three teachers' discourses, as well as changes across the four lessons for each teacher, were investigated. Reification and alienation in teachers' oral and written discourse were primarily visible in the word-use and endorsed narratives. Algebraic entities relevant in this analysis were concepts that potentially include algebraic symbols, for example, variable, equality, expression, equation and formula.

Table 1. Participants in the study.

	Teacher 1 (T1)	Teacher 2 (T2)	Teacher 3 (T3)
Years of teaching experience	22	10	10
Number of students in the class	18	30	20*
Length of lessons	39–41 min	42–60 min	30–57 min
Percent of lesson time spent on whole class instruction	49 %	50 %	21 %
Characteristics of the lesson	Pair work Use of worksheets Use of manipulatives	Small group discussions Use of textbook and worksheets	Individual work using textbook with tasks
Content of the lesson	Equations using boxes and beans** Growing patterns using matchsticks, proportional reasoning	Variable expressions, equalities and equation solving	Variable expressions, equalities and equation solving

*Two of the four lessons in this class were duplicated, with 10 students at a time present. For those lessons, the instruction of one group was included in the analysis.
** see Rystedt, Kilhamn and Helenius (2016) for a description of this activity.

Word-use is analyzed in line with Güçler (2013) and Johansson and Österholm (2019). It is considered process-oriented if teachers use active verbs and focus on processes and actions. If they instead use nouns and static relations, the word-use is considered reified. A similar approach is also used in Shinno and Fujita (2021). The extent to which an algebraic entity is separated from the self indicates the degree of alienation. For example: 'we name the variable *b*' is a processual word-use, tightly connected to the self being active in a process of naming, whereas in the phrase '*x* is a number you do not know anything about' the variable *x* is reified as being a number, but still not alienated. In the phrase 'Johan's age is *x*' the word-use indicates that the variable *x* is reified and alienated since it describes a state disconnected from the person speaking.

Visual mediators used in relation to algebraic entities are categorized as graphs, iconic/non-formal symbols (both concrete and drawn), or algebraic symbols (in line with Güçler, 2013; Shinno, 2018; Shinno & Fujita, 2021). In mathematics education literature, such mediators are usually referred to as representations.[1]

Narratives are recurring statements tightly connected to word-use. In this study, a narrative is considered to be endorsed by the teacher if it is used repeatedly, at least three times in different situations, in the teacher's discourse. Focus is on metalevel narratives, such as 'an expression is to describe something with variables' and 'an expression is a system of symbols', and not on object-level narratives such as formal definitions, theorems and facts related to the mathematical concepts (in line with Güçler, 2013; Shinno & Fujita, 2021). Regarding objectification, endorsed narratives are analyzed with respect to whether the algebraic entity is treated as a process or as an object, in the words, visual mediators and routines used in relation to the narratives.

Routines are defined by identifying recurrent patterns in the discourse. First applicability conditions are identified (i.e. what particular circumstances prompt the pattern to occur). Then, a specific course of action and a closing condition are identified (e.g. Nachlieli & Tabach, 2019). When a routine was identified, it was analyzed in terms of *when* and *how* it was used (Sfard, 2008). In particular, we paid attention to whether the routine focused on actions or on the underlying mathematical properties. In line with Caspi and Sfard (2012) and Nachlieli and Tabach (2012), a routine was considered action-oriented if it was marked with the use of verbs, and reified in accordance with the number of direct references to human actions.

6. Results

The algebraic content was sequenced in similar order by all three teachers over the four lessons, that is, first variables, then expressions and last equations. However, in what way and how explicitly the content was treated differed. For example, expressions were never handled as separate entities by T1, only introduced implicitly as left- and right-hand sides in equations. In relation to our RQ (characteristics of a more or less objectified algebra discourse), we first describe the use of visual mediators in the lessons, and then word-use, endorsed narratives and routines related to the object- or process-orientation of the discourse.

6.1. Use of visual mediators

An important difference between the three teachers' discourses was found in their use of algebraic symbols as visual mediators. Table 2 shows an overview of what kind of visual mediators the teachers used across the four lessons. In the first lesson, T1 used physical mediators, in the form of boxes with an unknown number of beans inside, which in the second lesson were replaced by iconic symbols, for example, 100• to denote 100 loose beans, and 25□ to denote 25 boxes with an unknown number of beans. In the third lesson, T1 introduced formal algebraic symbols, such as $4x + 10 = 2x + 8$, in which x was equated with the 'boxes'. It is notable that x was the only letter symbol used by T1. Both T2 and T3, on the other hand, introduced various formal algebraic symbols from the start, such as $a + 3$ and $4\,h$.

T2 also expressed relations in which expressions were named, for example, Arvin $= (x + 4)$ and soon narrowed this down to only using x as a visual mediator for variables. Although both T1 and T2 moved over the four lessons towards the convention of using x (Table 2), we could see that they followed different routes. T3 stayed with various alphanumeric symbols, although closely context-related. For example, '*c*' for circumference, and '*k*' for kamel (camel, see Appendix C). T3 also used informal symbols,

Table 2. Overview of the teachers' use of visual mediators.

Teacher	Visual mediators
T1	physical → iconic → formal (only x)
T2	formal (various) → formal (only x)
T3	formal (various)/non-formal

→ indicates a change over the course of the four lessons.
/ indicates that visual mediators existed interchangeably in the lessons i.e. no explicit change.

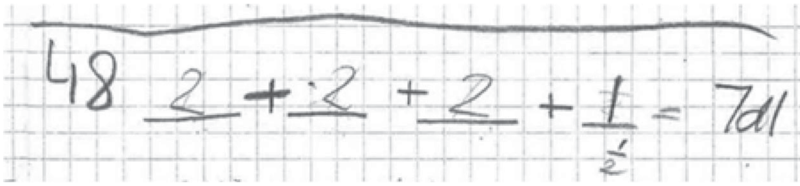

Figure 1. The teacher draws lines to represent number relations stated in the task formulation (task no. 48). The fourth line is half as long as the other three. A trial-and-error strategy is then used to find numbers that would fit these relations and add up to 7.

Table 3. Summary of the teachers' mathematical discourse over the lessons around variables, expressions and equations, as well as an overview, with respect to instances of objectification (or not).

	T1			T2			T3		
	Word use	Endorsed narratives	Routines	Word use	Endorsed narratives	Routines	Word use	Endorsed narratives	Routines
Variables	*P*	*P*	–	*P* → RA	PA	–	*P*	*P*	*P*
Expressions	*	*	*	*P* → RA	*P* → RA	R	R	R	–
Equations	*P*	*P*/R	*P*	*P*/R	*P*/R → RA	*P*	**	*P*/R	*P*
Overall	*P*	*P*→ *P*/R	*P*	*P* → RA	PA → RA	*P*	*P* → R**	*P* / R	*P*

Processual discourse (*P*), Reified discourse (R), Alienated discourse (A).
→ indicates a change over the course of the four lessons, / indicates that instances belonging to both types existed interchangeably in the lessons.
*Expressions were only implicitly dealt with, as part of an equation, never identified as a separate object.
**T3 was inconsistent in word-use, sometimes mixing up the terminology.

such as long and short lines (see Figure 1) to denote unknown quantities of different value in situations when the equations were not given.

The teacher draws lines to represent the number relations stated in the task formulation (task no. 48). The fourth line is half as long as the other three. A trial-and-error strategy is then used to find numbers that would fit these relations and add up to 7.

6.2. Process- or object-oriented discourse

Word-use, endorsed narratives and routines in the three teachers' discourses are summarized in Table 3. Discursive changes over the four lessons are indicated with an arrow. Overall, an object-oriented discourse, showing instances of reification (R) and alienation (A), was most prominent for T2, where both processes of reification and alienation could be identified, and a discursive change was visible over time. Examples from each teachers' lessons are given below the table to provide a comprehensive description of what types of discursive shifts were made possible (or not), as well as causes for the differences.

Concerning word-use and endorsed narratives, the greatest difference was found between T1 and T2. Whereas T2 showed many signs of reified and/or alienated discourse, these were rare in T1's discourse. For example, T2 started to describe a variable as an object 'you don't know anything about', then an object 'you decide a value for', which is an instance of reification but not alienation, and finally, the variable was treated as an alienated object that symbolizes an indeterminate quantity that can be a constituent of an expression. After the first introduction of variables, T2 introduced mathematical expressions as 'to describe something with variables' (a process connected to the doer: 'to describe'), then expanded the narrative to compare different expressions: 'various expressions can describe the same situation, for example $5 + x = x + 5$ and $4x = x + x + x + x$'. Later on, T2 talked about an expression as 'a system of symbols', which indicated that the mathematical expression was talked about as an alienated and reified object. T2 endorsed the narrative 'an expression is not an answer' and addressed the difference between writing and calculating an expression on a metalevel (see Appendix B, time 45:37). This development of the discourse over the four lessons provides an opportunity for discursive change from process-oriented to object-oriented.

T1, on the other hand, referred to variables as 'secret' or 'hidden' numbers, a narrative present in all four lessons. Although these words indicate objectification, they imply that the numbers are connected to the individual's experiences of 'finding' or 'revealing'. Thus, variables are not alienated. Furthermore, there was a focus over the lessons on the process of how this revealing is done (more thoroughly described below), which indicated a process-oriented discourse and limited opportunities for discursive change.

T3's discourse included several instances that could be regarded as reification (Table 3). For example, both the narratives 'expressions should be useful' and 'expressions are about you showing how to calculate something, to write this with letters and numbers without an equal sign', were used from the start. These narratives indicated that expressions are used to create some kind of mathematical structure, and connect expressions to the doer, which in turn indicates that the reification process had started, but that the discourse was not alienated. T3's discourse did not change much over the lessons. However, T3 was inconsistent in the words that were used for the different mathematical entities. For example, in the equation $F = E + 36$, the expression $E + 36$ was referred to as 'the variable', but in the equation $C = 8x$, the expression $8x$ was referred to as a 'formula'. When talking about equations, the words equation, formula, expression and variable were used interchangeably by T3, revealing inconstancies in word-use.

A similarity between all three teachers was their endorsement of the narrative that 'an equation is an equality', which indicated that all teachers handled equations as static relations. With respect to this narrative, there were aspects in the discourse in relation to equations that could be regarded as object-oriented. T2 also established the narratives that equations are 'a more compact way of … ' and that equations make it 'easier to see and keep track … ', which indicated that equations were treated as alienated objects.

Routines were mainly process-oriented in all three teachers' discourse, where focus was on the 'doing' rather than on the underlying mathematical structure (Table 3). One prominent routine for solving equations was based on the cancellation law. In T1's classroom, the particular solution procedure was introduced using concrete representations and the instruction to 'take away the same', thus limiting the routine to addition, which gave rise to difficulties when the students met equations involving subtraction (see Appendix A). An interesting result from T3's classroom was that a routine that was introduced for solving equations based on cancellation was limited to situations where the equations were given from the start. When the task included constructing an equation, and then solving it, another routine based on trial and error was present (described in Figure 1).

One exception to process-oriented routines was in T2's discourse concerning expressions (Table 3). The routine started with a relation involving an unknown; the course of action was then to describe this relation with formal algebraic symbols, including an alphanumeric symbol for the variable. The closing condition for the routine was when 'a system of symbols' had been created. This routine indicated a focus on the created object, the system of symbols, and thus can be considered as contributing to a reified discourse (i.e. object-oriented). Because the routine was connected to the 'doer', the alienation process was not considered to be present.

6.3. Characteristics of a more or less objectified algebraic discourse

In summary, the analyses showed that the most objectified discourse (T2) included a consistent and varied use of formal algebraic symbols as visual mediators, generalizable routines and a notable progression from processual to objectual over the four lessons of introduction to symbolic algebra. Whereas a less objectified algebraic discourse (T1 and T3) included a considerable use of informal symbols as visual mediators and routines that were limited to specific situations, and sometimes also inconsistencies in word use.

7. Discussion and conclusions

The purpose of this study is to explore potential obstacles for algebra learning from a commognition perspective. From this study's ontological point of view, learning mathematics presupposes reification and alienation. Consequently, what a person is capable of thinking is determined by the discourse a person has developed through interaction with others (Sfard, 2008, 2020). Previous studies have shown that students' algebraic learning follows a process-to-object trajectory (Blanton et al., 2017; Caspi & Sfard, 2012). A basis for this study is that the discourse of a classroom delimits what is possible for students to learn since learning is defined as a change of discourse (Sfard, 2008, 2016) and that language issues are important in mathematics learning (Erath et al., 2021). We have in detailed analyzed the available discourses in three Grade 6 algebra classrooms. Our results show aspects of objectification in all three teachers' discourses, although the variation was quite large. Below we discuss our results in terms of students' opportunities to develop an objectified algebra discourse.

Objectification was most prominent in T2's discourse, with a notable progression in the process of objectification as the objects became alienated from the actor in consecutive lessons. This kind of progression in the objectification process could be regarded as essential in the mathematical discourse that teachers model (implicitly or explicitly), in line with what is shown in Shinno (2018). A lack of objectification, on the other hand, as seen in T1's discourse resembling results in Sfard (2016), may limit students' opportunities to discursively turn processes into objects.

Opportunities to engage in a new type of discourse help students to develop new ways to communicate about algebraic objects through reification and alienation, which is fundamental for their algebra learning (Sfard, 2008). At the same time, simply being immersed into a new discourse may not be enough. It has previously been shown that students benefit from reflecting upon the discursive elements and examining, on a metalevel, how aspects of the discourse changes (cf. Emre-Akdoğan et al., 2018; Güçler, 2016; Nachlieli & Tabach, 2012; Shinno, 2018). In our study, none of the teachers explicitly addressed metalevel aspects of the discourse. For example, when it came to routines, focus was on the *doing*, without addressing underlying mathematical properties. We argue that this will constrain students to rely on imitation, and limit their opportunities to generalize and overcome mathematical difficulties (cf. Güçler, 2016). For example, the routine for solving equations by 'taking away' the same number of physical boxes or beans from each side, which was used in T1's classroom, is only useful for addition. Furthermore, the use of physical mediators in this routine has been shown to restrict students' ability to solve equations to certain contexts (Rystedt, Kilhamn & Helenius, 2016). While most of the earlier studies concerned more advanced mathematics and older students, we have studied the very first lessons of introducing formal algebra in Grade 6. Since raising students' language awareness is described by Erath et al. (2021) as a major design principle for enhancing language for mathematics learning, a question for further research is whether it would be fruitful to impose metalevel reflections from the very start, in the transition from informal to formal algebraic symbolism.

Furthermore, it is previously noted that explicitly addressing metalevel aspects is easier to accomplish in an alienated discourse that has developed gradually and consistently (cf. Güçler, 2016; Nachlieli & Tabach, 2012). As our results show, a major difference between a more or less objectified (reified and alienated) discourse was the use of formal algebraic symbols as visual mediators, including both the type of symbols that were used and how consistently they were used. In order to generalize, representations need to mediate variables as mathematical objects and not as real objects involving some unknown value. The difference in the trajectories we see in the discourses of T1 and T2 is important. Both end up using x as a visual mediator for a variable, often for a value that is as yet unknown. For T1, x is a symbol used instead of a square, which is an iconic symbol used instead of a real box containing a specific, albeit unknown number of beans. In this context, x can only be a natural number. Although it is possible in that discourse to talk about an unknown value x, a discursive change is required if the student is to develop their algebraic thinking, since the variable is not yet reified as a mathematical object. In T2's discourse, x is a visual mediator that could essentially mediate any kind of unknown value. Initially, many different letters were used, but later x was introduced as a general symbol to be utilized instead of any other letter standing for an unknown value. In that discursive shift, the arbitrary nature of the symbol comes to the fore. Furthermore, relating symbols to physical objects, for example, camels or people, as we saw in T3's discourse, has previously been highlighted as an obstacle for algebraic learning because students focus on the object and not on the number the symbol represents (Arcavi et al., 2017; Blanton et al., 2017; Caspi & Sfard, 2012). Although formal alphanumeric notation may not be necessary at the start, its strength in supporting a discursive shift should not be ignored. Thus, we argue that teachers' different use of visual mediators is a major reason for the observed differences in the alienation process. Similar results are found by Johansson and Österholm (2019), where teachers' discourses were more objectified when symbols for mathematical concepts had been introduced, and by Caspi and Sfard (2012) who discuss that alphanumeric symbol system enables and amplifies objectification. Although the 'early algebra' movement has been beneficial in terms of introducing algebraic thinking in the early years without burdening students with the syntax of formal symbolism (Blanton et al., 2017; Kaput et al., 2008; Kieran, 2018; Radford, 2014), our results indicate that informal representations are more connected to processual word use and context-specific routines. In addition to teaching prospective teachers about early algebra, as emphasized by Hohensee (2017), we also recommend that the role played by formal algebraic symbols in the transition to a more object-related discourse is explicitly brought to attention. We conclude that a conscious and clear use of formal algebraic symbols is beneficial for students' objectification process.

A similar conclusion relates to the importance of conscious and consistent word-use. Our results showed at a first glance a puzzling word-use in T3's discourse. But after detailed analysis it became clear that it was a matter of inconsistency in the teacher's word-use. As discussed by Nachlieli and Tabach (2012), teachers need to use correct words in a consistent way in order to support students' discursive development in a new mathematical discourse. We urge teacher educators and teacher professional development projects to emphasize precision in word-use so that teachers become more secure about which words to use, as well as when and how. Furthermore, using the terms 'secret number' and 'hidden number' that we saw in in our analyses, are common when talking about unknowns and variables. The fact that a number is secret (someone knows it but not you) or hidden (it is there, but out of sight for you), indicates that the number is specific and real, rather than varying and abstract. Although easy for a child to relate to, they are clearly not alienated. Since mathematical objects are abstract, the words used are often metaphorical. Even if such words are useful when introducing the algebraic idea of talking about and operating on numbers that are not known or specified, a discursive shift is still needed, incorporating words that represent something as yet unspecified or a value that can vary while the expressed relationship stays the same (cf. Steinweg et al., 2018). We argue that teachers should be careful not to restrict their word-use to words that connect to the individual or to a real object, but instead to introduce words like unknowns, placeholders and variables. In algebra, these terms have distinct meanings, as described by Ely and Adams (2012), and to distinguish between them the discourse needs to pass beyond secret and hidden numbers. This discursive shift from pre-algebraic to algebraic terminology is an important issue for both teacher education (Hohensee, 2017) and teachers' professional development (Kilhamn, 2014).

An interesting reflection is that all teachers explicitly talk mainly about variables and equations, but not much about expressions. Only one teacher (T2) is consistent in her handling of expressions and explicitly addresses expression as an entity in its own right, which also paves the way for a visible discursive objectification process. A question to ask is whether it is important for students' algebraic understanding to recognize an expression as a separate mathematical object, or if it is sufficient to treat expressions implicitly when discussing formulas and equations. For example, how do students understand 'Arvin $= (x+4)$', and do they see this as different from '$F = E + 36$'? In the first case, focus is on the expression $x + 4$ as a representation of Arvin's age, whereas in the second, focus is on the relation between the numbers represented by F and E. We know from the review of the literature that there exists an abundance of research concerning the teaching and learning of variables and equations, but very little focusing on expressions. This is an area that needs to be addressed in the future.

In this article, we set out to examine characteristics of a more or less objectified algebraic discourse in the introduction of school algebra in middle school (RQ). As discussed above, we identified both similarities and differences. For example, when visual mediators were used, physical or iconic visual mediators limited the alienation process, whereas arbitrary symbols seemed to promote reification and eventually alienation. Furthermore, the use of words that related to an individual or a physical object, as well as inconsistent word-use were other aspects that seemed to be connected to a lack of objectification. Since teachers model the classroom discourse (explicitly or implicitly), a lack of objectification in the teacher's own discourse may limit opportunities for students' learning (Nachlieli & Tabach, 2012; Sfard, 2016; Tabach & Nachlieli, 2016). Finally, in view of previous research emphasizing the importance of addressing metalevel aspects (Emre-Akdoğan et al., 2018; Güçler, 2013; 2014), the failure to address metalevel aspects in these three classrooms could restrain students' opportunities to develop an objectified discourse.

Our data does not say anything about students' actual discursive changes in relation to the teachers' discourse. Dealing only with how a teacher models a discourse does not reveal in what way the students appropriate the discourse, therefore this relation will need to be addressed in future studies. Although another theoretical perspective might have provided other relevant results, our findings contribute to the current literature by broadening the field's understanding of middle-school teachers' algebra teaching and the discourses that emerge in their classrooms as they approach a formal symbolic algebra, and the impact this may have on students' learning from a commognitive perspective.

Note

1. A representation is often described as something that stands 'instead of' a mathematical object. Describing a representation as a visual mediator neither agrees with nor contradicts that view. A visual mediator is a sign of some kind that gives a person visual access to a mathematical object.

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