Mathematical Reasoning Requirements in Swedish National Physics Tests

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MATHEMATICAL REASONING REQUIREMENTS IN SWEDISH NATIONAL PHYSICS TESTS

This paper focuses on one aspects of mathematical competence, namely mathematical reasoning, and how this competency influences students' knowing of physics. This influence was studied by analysing the mathematical reasoning requirements upper secondary students meet when solving tasks in national physics tests. National tests are constructed to mirror the goals stated in the curricula, and these goals are similar across national borders. The framework used for characterising the mathematical reasoning required to solve the tasks in the national physics tests distinguishes between imitative and creative mathematical reasoning. The analysis process consisted of structured comparisons between representative student solutions and the students' educational history. Of the 209 analysed tasks, three-fourths required mathematical reasoning in order to be solved. Creative mathematical reasoning, which in particular involves reasoning based on intrinsic properties, was required for one third of the tasks. The results in this paper give strong evidence that creative mathematical reasoning is required to achieve higher grades on the tests. It is also confirmed that mathematical reasoning is an important and integral part of the physics curricula; and, it is suggested that the ability to use creative mathematical reasoning is necessary to fully master the curricula.

Keywords: Mathematical reasoning, Creative mathematical reasoning, Upper secondary school, Physics tests, Swedish national assessment.

Introduction

Mathematics and physics are historically closely intertwined, and many mathematical concepts have been developed to describe the laws of nature. How this relationship becomes apparent in a school context and how it might affect students' learning have been discussed from different points of view in educational research. Some of the discussions focus on how physics can influence the learning of mathematics, referred to below as *physics in mathematics.* Other discussions focus on the learning of physics and are concerned with various aspects of its relation to mathematics, and this is referred to as *mathematics in physics*.

Physics in Mathematics

Blum and Niss (1991) discuss the great value of maintaining a close relationship between mathematics and physics in school because physics can provide good examples for validating mathematical models. In a paper by Doorman and Gravemeijer (2008), the authors discuss the advantage of learning mathematical concepts through mathematical model building and how examples from physics allow for a better understanding of the concepts. Hanna and Jahnke (2002) refer to e.g. Polya (1954) and Winter (1978) when they discuss the advantage to use arguments from physics in the teaching of mathematical proofs. The importance of using physics to facilitate students' learning of various mathematical concepts is also discussed by Marongelle (2004), who concludes that using events from physics can help students to understand different mathematical representations.

Mathematics in Physics

Tasar (2010) discusses how a closer relation between the school subjects of mathematics and physics can contribute to the understanding of physics concepts and can help ensure that students already understand the mathematical concepts needed in physics. Similar suggestions are done by Planinic et al. (2012), who in their study of high school students' success on parallel tasks in mathematics and in physics concluded that students' knowledge is very compartmentalized and that stronger links between the mathematics and physics education should be established. According to Basson (2002), a closer relationship might also decrease the amount of time physics teachers spend on redoing the mathematics students need in physics. Michelsen (2005) discusses how interdisciplinary modelling activities can help students to understand how to use mathematics in physics and to see the links between the two subjects. Redish and Gupta (2009) emphasize the need to understand how mathematics is used in physics and to understand the cognitive thinking of experts in order to teach mathematics for physics more effectively to students. Basson (2002) mentions how difficulties in learning physics not only stem from the complexity of the subject but also from insufficient mathematical knowledge. Bing (2008) discusses the importance of learning the language of mathematics when studying physics. Nguyen and Meltzer (2003) analysed students' knowledge of vectors and concluded that there is a gap between students' intuitive knowledge and how to apply their knowledge in a formal way, which can be an obstacle when learning physics. Tuminaro (2002) analysed a large body of research, and categorized studies concerning students' use of mathematics in physics according to the researchers approach to the area. The four categories are: (i) the observational approach; (ii) the modelling approach; (iii) the mathematical knowledge structure approach, and (iv) the general knowledge approach.

Mulhall and Gunstone (2012) describe two major types of physics teacher, the *conceptual* and the *traditional.* Mulhall and Gunstone conclude that a typical teacher in the conceptual group presumes that students can solve numerical problems in physics without a deeper understanding of the underlying physics theories. A typical opinion among teachers in the traditional group is that physics is based on mathematics and that a student develops an understanding of the physics by working with numerical problems. Doorman and Gravemeijer (2008) notice (with reference to Clement, 1985 and Dall'Alba et al., 1993) that most of the attention in both physics and mathematics is on the manipulations of formulas instead of focusing on why the formulas work.

Learning Physics

When discussing learning in physics, there is, of course, a large body of additional literature that is relevant to consider depending on what questions one is studying. A lot of research about teaching and learning physics has been conducted by what Redish (2003) refers to as the Physics Education Research (PER) community. When studying how individuals learn physics, certain cognitive principles have to be considered (Redish, 2003). This approach is discussed by diSessa (e.g. in 2004), who emphasises the micro levels but from a knowledge-in-pieces perspective. This perspective is not restricted to the learning of physics, but is also applicable in mathematics. According to this micro-perspective, there are many different levels at which a concept can be understood, and contextuality has to be taken into consideration. Thus, in order to understand a student's learning, his or her understanding of a particular concept has to be studied in a variety of different contexts (diSessa, 2004).

Mathematics in the Syllabuses

The upper secondary school in Sweden is governed by the state through the curriculum and the syllabuses. During the last decades there has been a gradual change toward a stronger focus on process goals, and they are present in the curriculum from 1994 (Swedish National Agency for Education (SNAE), 2006). These shifts are influenced by and similar to international reforms that aim at enriching both mathematics and physics. Content goals are complemented with process goals as those in the NCTM Standards (National Council of Teachers of Mathematics, 2000), and in the NGSS (Next Generation Science Standards Lead States, 2013) where it e.g. is explicated that "emphasis is on assessing students' use of mathematical thinking and not on memorization and rote application of problem-solving techniques" when high school students use mathematics in physics (NGSS, 2013, HS-PS1-7, Matter and its Interactions). In the framework for PISA 2009 it is emphasized to focus on the mastery of processes and the understanding of concepts (OECD, 2009), and in the TIMSS framework the thinking process is explicated as one of the two dimensions to be assessed

(Garden et al. 2006). For a more comprehensive discussion about the reforms and their backgrounds see e.g. Boesen et al. (2014, pp. 73-74). A central part of the reforms concerns reasoning and its central role in problem solving and in the individual's development of conceptual understanding

In the Swedish syllabuses the aims and objectives of each specific course are detailed and it is indicated what knowledge and skills students are expected to have acquired upon completion of the various courses. According to the general syllabus in physics, the teaching should aim to ensure that the students e.g. "develop their ability to quantitatively and qualitatively describe, analyse and interpret the phenomena and processes of physics in everyday reality, nature, society and vocational life and to develop their ability with the help of modern technical aids to compile and analyse data as well as simulate the phenomena and processes of physics"[1] (SNAE, 2000). Mathematics is thus explicitly required when making quantitative descriptions of phenomena and implicitly required when analysing data. In the particular syllabuses for the two courses Physics A and Physics B, mathematics is mentioned more explicitly. Physics A is a prerequisite for Physics B, and in the latter course there are higher demands both on the mathematical processing and on the conceptual understanding of physics phenomena (SNAE, 2000).

The literature review shows that there is a significant amount of educational research on the relation between the school subjects of mathematics and physics that support the necessity of different mathematical competencies when learning physics. However, no studies on what type of mathematical reasoning (see Theoretical framework) is required of physics students were found. The impact of mathematical reasoning *on mathematical* learning has been discussed and studied from multiple perspectives. Schoenfeld (1992), for example, points out that a focus on rote mechanical skills leads to poor performance in problem solving. Lesh & Zawojeskij (2007) discuss how emphasising low-level skills does not give the students the abilities needed for mathematical modelling or problem solving, neither to draw upon interdisciplinary knowledge. Students lacking the ability to use creative mathematical reasoning thus get stuck when confronted with novel situations, and this hamper their possibilities to learn (Lithner, 2008). Since mathematics is a natural part of physics, it is reasonable to assume that the ability to use mathematical reasoning is an integral part of the physics knowledge students are assumed to achieve in physics courses. Therefore, it should be desirable to get a picture of the mathematical reasoning requirements students encounter and need in order to master or fully master the physics curricula.

Theoretical Framework

The definition of mathematical reasoning and the framework that is used for the analyses in this paper were developed by Lithner (2008) through empirical studies on how students engage in various kinds of mathematical activities. The initial purpose of Lithner's studies was to analyse students' rote thinking and how this may lead to learning difficulties in mathematics. As a result, *reasoning* was defined as "the line of thought adopted to produce assertions and reach conclusions in task solving" (Lithner, 2008, p. 257). *Mathematical reasoning* is used as an extension of a strict mathematical proof to justify a solution and is seen as a product of separate reasoning sequences. Each sequence includes a choice that defines the next sequence, and the reasoning is the justification for the choice that is made. The mathematical foundation of the reasoning can either be *superficial* or *intrinsic*. The accepted mathematical properties of an object are of different relevance in different situations. This leads to a distinction between *surface* properties and *intrinsic* properties, where the former have little relevance in the actual context and lead to superficial reasoning and the latter are central and have to be taken into consideration in the given context (Lithner, 2008, pp. 260 – 261). Depending on whether this reasoning is superficial or intrinsic, the framework distinguishes between *imitative reasoning* and *creative mathematical founded reasoning*. The framework has been used in previous studies to categorise tasks according to mathematical reasoning (e.g. Palm, Boesen & Lithner, 2011) or to categorise actual students' mathematical reasoning in problematic situations (e.g. Sumpter, 2013).

Creative mathematically founded reasoning

Creativity is an expression often used in different contexts and without an unequivocal definition (for a discussion see Lithner (2008, pp. 267-268)). Creativity within the framework that is used in this paper takes the perspective of Haylock (1997) and Silver (1997) in which creativity is seen as a thinking process that is novel, flexible, and fluent. *Creative mathematical reasoning*[2] *(CR)* fulfils all of the following criteria: "**i. Novelty**. A new reasoning sequence is created or a forgotten one is recreated. **ii. Plausibility**. There are arguments supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible. **iii. Mathematical foundation**. The arguments made during the reasoning process are anchored in the intrinsic mathematical properties of the components involved in the reasoning." (Lithner, 2008, p.266).

Imitative Reasoning

Imitative reasoning is categorised as *memorised reasoning (MR)* or *algorithmic reasoning (AR)*. The arguments for the chosen solution method (i.e. the reasoning) can be anchored in surface mathematical properties. "MR fulfils the following conditions: **i.** The strategy choice is founded on recalling a complete answer. **ii.** The strategy implementation consists only of writing it down." (Lithner, 2008, p. 258).

If some kinds of calculations are required to solve the task, there is often no use in remembering an answer. Instead it is more suitable to recall an algorithm. The term algorithm is used here in a broad sense and refers to all the procedures and rules that are needed to reach the conclusion of a specific type of task, not just the calculations required to reach a conclusion. "AR fulfils the following conditions: **i.** The strategy choice is to recall a solution algorithm. The predicted argumentation may be of different kind, but there is no need to create a new solution. **ii.** The remaining parts of the strategy implementation are trivial for the reasoned, only a careless mistake can lead to failure." (Lithner, 2008, p.259).

Depending on the argumentation for the choice of the used algorithm, AR can be subdivided into the three different categories of *familiar algorithmic reasoning (FAR),* delimiting algorithmic reasoning and *guided algorithmic reasoning* e.g. *text-guided (GAR)* or person-guided. In this study, only the categories of FAR and GAR are used. FAR fulfils: "**i.** The reason for the strategy choice is that the task is seen as being of a familiar type that can be solved by a corresponding known algorithm. **ii.** The algorithm is implemented." (Lithner, 2008, p. 262). GAR fulfils: "**i.** The strategy choice concerns identifying surface similarities between the task and an example, definition, theorem, rule or some other situation in a text source. **ii.** The algorithm is implemented without verificative argumentation." (Lithner, 2008, p.263).

Local and Global Creative Mathematical Reasoning

Lithner (2008) introduces a refinement of the category (CR) into *local CR (LCR)* and *global CR (GCR)* that captures some significant differences between tasks categorised as CR. This subdivision has been further elaborated by other scholars, e.g. Boesen, Lithner and Palm (2010) and Palm et al. (2011). In LCR, the reasoning is mainly MR or AR but contains a minor step that requires CR. If instead there is a need for CR in several steps, it is called GCR, even when some parts contain AR and/or MR.

Non-mathematical Reasoning

The analytical framework in this paper introduces an additional category called *non-mathematical reasoning (NMR)*. This consists of those tasks that can be solved by using just a knowledge of physics. Physics knowledge here refers to relations and facts that are discussed in the syllabuses and textbooks of the physics courses but not in the mathematics courses, for example, the fact that angle of incidence equals angle of reflection. In the same way, the concept of mathematics refers to school mathematics that is introduced in mathematics courses for students at upper secondary school or the mathematics assumed to already be known according to the curricula.

Research Question

By analysing the mathematical reasoning required to solve tasks in national physics tests, the idea is to capture the mathematical reasoning that is required to master or fully master the physics curricula. It is explicated in the physics syllabuses that the use of mathematics is incorporated in the goals and that the national tests are the government's way of concretising the physics curricula. Based on the definitions in the definitions of the theoretical framework described above, the following research questions were asked:

- Is mathematical reasoning required of upper secondary students to solve national physics tests from the Swedish national test bank?
- If mathematical reasoning is required, what is the distribution of physics tasks requiring CR compared to tasks that are solvable with IR?

Physics Tests from the National Test Bank

About 12% of all students in upper secondary school in Sweden are enrolled in the Natural Science Programme or the Technology Programme (SNAE, 2011). In both programmes, the course Physics A is compulsory whereas the more advanced course–Physics B–is elective. The aim of the physics courses is that the students should attain various goals specified in the syllabuses. Written tests are commonly used as an assessment of the students' achievements, and a student's grade in a course depends on how well the student has achieved the goals for the course (SNAE, 2000). The descriptions of the goals and the different grade levels are quite brief in the syllabuses, and the intention is that the syllabuses and curriculum should be processed, interpreted, and refined locally in each school. In order to accomplish equivalent assessment in physics, the SNAE provides assessment supports, including the National Test Bank in Physics. In this way, the physics tests can be considered as the government's concretisation of the syllabuses for physics. The character and design of the tasks in the tests stress what is covered in the taught curriculum. The tests also influence the teachers' interpretation of the syllabuses, which by extension stresses what the students focus on (Ministry of Education and Research, 2001; SNAE, 2003).

The material in the National Test Bank is classified and can be accessed via the Internet only by authorised users. The material consists of single tasks to choose from or complete tests that comprise the goals for Physics A or Physics B. In total, there are 847 tasks to choose from and 16 complete tests for each of the Physics A and Physics B courses, all classified. The first tests are from 1998 and the latest is from spring 2011. Besides the classified examples, there are five tests for each course that are open for students to practice on. These give the students an idea of what the tests look like and what is required when taking a test (Department of Applied Educational Science, 2011). As opposed to national tests in mathematics, the teachers are not obligated to use the test from the National Test Bank in physics. However, it is common that these tests are used as a final exam in the end of the physics courses (SNAE, 2005)**.**

Since the beginning of the national testing program, there has been a change in the design of the tasks on the tests. In the beginning, there was more or less only one correct solution to each task. This has evolved into a higher degree of open tasks that can be solved using different approaches. For the past ten years, the final task has been an "aspect-task" that is assessed according to the achieved level in different assessment groups. These aspect-tasks include initial parts that are easily accessible for most students and parts that are a challenge designed for more proficient students. The task is designed to be easy to start with, but it should also include a challenge to more proficient students. The first three years of the testing program (1998 – 2000), there was an experimental part included in the tests, but this part is not included in the analysis in this study.

Method

This study analysed the December 1998, May 2002 December 2004, May 2005, and December 2008 tests for the Physics A course and the May 2002, May 2003, May 2005, February 2006, and April 2010 tests for the Physics B course. The first tests chosen were the unclassified tests so that examples could be discussed in the present article. These tests are unclassified by the National Educational Agency to serve as representative interpretations of the syllabuses and the curriculum. To have five tests from each course, the remaining tests were randomly selected among the classified tests.

Categorisation of Mathematical Reasoning Requirements

To categorise physics tasks according to reasoning requirements, solutions to respective task are required. Whether a task is solvable by IR or if the solution requires CR depends on the educational history of the solver (in this case the test-taker) (cf. Björkqvist, 2001; English & Sriraman, 2010; Lesh & Zawojewski, 2007; Schoenfeld, 1985; Wyndhamn et al., 2000.). The required reasoning refers to what kind of reasoning is sufficient to solve a task, and the framework described above allows for a determination of this.

The solutions used in the analysing procedure were constructed by the researcher. These solutions were determined to be plausible student solutions based on the researcher's experience as a physics teacher together with access to the solution manuals in which proposed solutions are given. Some of the solutions in the manual are authentic student solutions because several of the tasks had been tested on real students before the tasks were included in a test.

Because no students were involved in the present study, there were no actual learning history to consider. Studies on mathematics education suggest that most of the learning activities consist of students working with their textbooks (SNAE, 2003). In an evaluation of physics education in lower secondary school, it was found that the teaching is guided by the textbooks (Swedish Schools Inspectorate, 2010). In addition, The Ministry of Education and Research (2001) has discussed the fact that textbooks and assessments are seen as two of the most important tools in mathematics education. In a qualitative study of a physics class, Engström (2011) showed that the textbook still plays an important role in guiding the education, and the TIMSS Advanced 2008 report showed that teachers mostly use the textbook in physics courses to choose and solve problems from (SNAE, 2009a). Based on the findings described above, the students' prior knowledge in physics and mathematics in this study was equated with the content of the textbooks used in their courses. There are, of course, other factors that play a part in individual students' previous experience, including tasks discussed during classes and/or experience of physical principles outside the classroom. The simplification used in this study was necessary due to the complexity of students' educational history and was reasonable according to the discussion above.

This study considered textbooks in both mathematics and physics. When taking the tests, the students are allowed to use a handbook designed for the physics courses in upper secondary school. The access to formulas and definitions in this handbook had to be taken into account when analysing the tasks in this study. The textbooks and the handbook were chosen among the books commonly used in the physics courses in upper secondary school. The books used for categorisation of the tasks in Physics A tests were "Ergo Fysik A" (Pålsgård et al., 2005a) and "Matematik 3000 Kurs A och B" (Björk & Brolin, 2001). For tests in Physics B "Ergo Fysik B" (Pålsgård et al., 2005b) and "Matematik 3000 Kurs C och D" (Björk & Brolin, 2006) were used. The handbook chosen was ”Tabeller och formler för NV- och TE- programmen" (Ekbom et al., 2004). Even if not all students in the Swedish upper secondary school are using the books above, they are a reasonable assumption for the education history of the average student. The procedure for analysing the tasks was given by the chosen framework, and an analysis sheet was used to structure the procedure. The steps comprised in the procedure are outlined in Figure 1 below and are used earlier in e.g. Palm et al. (2011).

Step I. Analysis of the assessment task – Answers and solutions	**Step II. Analysis of the assessment task – Task variables**
The first step in the procedure consisted of constructing a plausible student solution. The solution was then looked at from a mathematical perspective and categorised according to relevant mathematical subject areas that were required for the solution, e.g. asking if the solution included working with formulas, algebra, diagrams, solving equations, etc. Tasks with solutions not including any mathematical object were identified and categorised as NMR tasks. This categorisation is an addition to the original procedure used in previous studies. Mathematical objects refer to entities to which mathematics is applied. The first step also includes the identification of 'real-life' events in the task formulation. This identification is relevant because a described situation in the task could give a clue to a known algorithm that solves the task (see the Weightlifter (a) example below).	The next step in the procedure was to analyse the solution according to different task variables. The first variable was the explicit formulation of the assignment. The second variable was *what* information about the mathematical objects was given explicitly in the task compared to what information the students need to obtain from the handbook or that they have to assume in order to reach a solution. The third task variable concerns *how* the information was given in the task, e.g. numerically or graphically or whether it was interwoven in the text or explicitly given afterwards. The task could also include keywords, symbols, figures, diagrams, or other important hints the student can use to identify the task type and which algorithm to use. These features were gathered into the fourth task variable.
Step III. Analysis of the textbooks and handbook – Answers and solutions	**Step IV. Argumentation for the requirement of reasoning**
The third step in the analysis process focused on the textbooks and the handbook. Formulas used in the solution algorithm were looked for in the handbook, and the available definitions were compared to the constructed solution to the task. The textbooks were thoroughly looked through for similar examples or exercises that were solved by a similar algorithm. The theory parts in the text-books were also examined to see whether they contained any clues as to solve the task.	In the final step, the researcher produced an argument, based on steps I to III, for the categorization of the reasoning requirement for every task. In order to be categorised as FAR, there must have been at least three tasks considered as similar in the textbooks. It could then be assumed that the students will remember the algorithm, which might not be the case if there are fewer occasions. Three similar tasks was found to be an appropriate number in the study by Boesen et al. (2010). If the task was similar to a formula or definition given in the handbook, it was assumed that the student could use this as guidance in order to solve the task. Thus only one similar and previously encountered example or exercise was required for tasks categorised as requiring GAR. To be categorised as requiring MR, tasks with the same answer or solution should have been encountered at least three times in the textbooks. It was then assumed that the student could simply write the same answer on the test. If none of the above reasoning types were sufficient for solving the task and there was a need to consider some intrinsic mathematical property, the task was categorised as requiring some kind of CR.

Figure 1. Detailed outline of the analysis procedure

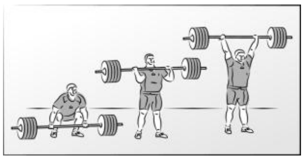
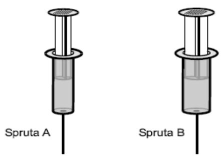
Validity and Reliability

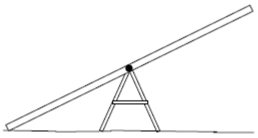
The resulting categorisation of tasks, theoretically established according to the procedure above is only meaningful if it represents the reasoning actually used by students while solving the tasks. Meaningful representation can be achieved with the well-documented criteria required for each category along with a routine for agreement and discussions about the categorisation. Higher reliability could also be reached with a less complex phenomenon, e.g. by defining creative mathematical reasoning as solutions consisting of more than three steps. However, this would give a low validity for the meaning of creative mathematical reasoning.

The validity of the analysis is dependent both on the appropriateness of the procedure used for the categorisation and how closely the categorisations resemble students' actual reasoning. The appropriateness is argued for above, and an argument for concordance is based on results from a study by Boesen et al. (2010). In that study, real students' actual mathematical reasoning used to solve tasks on mathematics tests were compared to the theoretically established reasoning requirements for the same tasks based on the same procedure that was used in the present study. It was shown that only 3 % of the tasks were solved with less creative reasoning than what was judged to be required, and 4 % of the tasks were either solved with more creative reasoning or not solved at all. These results indicate that the categorisation of reasoning as described here provides meaningful results. The construction of a plausible student solution is one of the four steps in the analysing procedure, and the author's experience with physics students and physics tests can be considered similar to the experience with mathematics students and mathematics tests in Boesen et al. (2010). Therefore, previous results demonstrating the method's validity can be considered to be valid for the present study. The categorisation of all tasks in the present study was made by the author. During the analysis process, there were tasks where the categorisation was straightforward and tasks where the categorisation could be considered as border-line cases. Typical examples of the different kinds of categorisation were continuously discussed in a reference group consisting of a mathematics education researcher well familiar with the analysis procedure and a mathematician. All difficult categorisations were discussed in the group, thus no inter-reliability estimate was calculated.

Data and Analyses

The tasks in Figure 2 were chosen to represent and illustrate the different types of analysis and the categorisations of the physics tasks. The idea was that the required reasoning would be represented by the constructed solutions. All of the tasks are chosen from publicly available national tests. Normally, subtasks are treated separately because the task variables and the analysis of the textbooks can be different. The first three tasks are examples where no hesitation concerning the categorisations occurred. The three subsequent examples are of tasks were the analysis was not as straightforward. The analyses are displayed in detail in Figure 3a to 3d.

Task 1 (“The Weightlifter (a)”)	**Task 2 (“The Weightlifter (b)”)**
A weightlifter is lifting a barbell that weighs 219 kg. The barbell is lifted 2.1 m up from the floor in 5.0 s.  a) What is the average power the weightlifter develops on the barbell during the lift?	A weightlifter is lifting a barbell that weights 219 kg. The barbell is lifted 2.1m up from the floor in 5.0 s. b) What is the average power the weightlifter develops on the barbell when he holds it above his head for 3.0 s?
Task 3 (“The Syringes”).	**Task 4 (“Charges on a thread”)**
A patient is going to get an injection. The medical staffs are reading in the instructions that they are supposed to use a syringe that gives the lowest pressure as possible in the body tissue. Which of the syringes A or B shall the staff choose if the same force, F, is applied and the injection needles have the same dimensions?” *Argue for the answer* 	In order to determine the charge on two small, light silver balls, the following experiment was conducted. The balls, which were alike, weighed 26 mg each. The balls were threaded onto a nylon thread and were charged in a way that gave them equal charges. The upper ball levitated freely a little distance above the other ball. There was no friction between the balls and the nylon thread. The distance between the centres of the balls was measured to 2.9 cm. What was the charge on each of the balls?” 
Task 5 (“The seesaw”)	**Task 6 (“Walking in water”)**

<table>
<tr>
<td>How can Lars, 70 kg, and his son Anton, 28 kg, place themselves on a 3.5 m long seesaw so that it stays in equilibrium?
</td>
<td>You are walking out into the water at a beach with a stony bottom. In the beginning, it hurts very much under your feet when you are walking on the stones, but when the water gets deeper it starts to feel less. When the water reaches you up to your chest, the stones do not feel painful anymore. Explain this.
Svar: ____________________
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____________________</td>
</tr>
</table>

Figure 2. Six examples of the tasks that were analysed.

	I. Analysis of the assessment task- Answers and solutions	II. Analysis of the assessment task- Task variables
Task 1	A typical solution from an average student could be derived by the relation between power and the change of energy over a specific period of time. In this task, the change in energy is the same as the change of potential energy for the barbell. Multiply the mass of the barbell by the acceleration of gravity and the height of the lift and then divide by the time to get the power asked for. The mathematical subject area is identified as algebra, in this case working with formulas. The identification of the situation to lift a barbell can trigger the student to use a certain solution method and is, therefore, included in this analysis as an identified "real-life" situation.	The assignment is to calculate the average power during the lift. The mass of the barbell, the height of the lift, and the time for the lift are all considered as mathematical objects. As mentioned above, an object is the entity one is doing something with. In this example, all of the objects are given explicitly in the assignment in numerical form. In the presentation of the task, there is also an illustrative figure of the lift.
Task 2	It is not necessary to use any mathematical argumentation in order to solve this task, and solution can be derived on physical reasoning alone. There is no lifting and, therefore, no work is done, and this means that no power is developed. This task is a typical example of an analysis resulting in the NMR categorisation.	Not a step to consider as this task is categorized as NMR
Task 3	To solve this task, the student can use the relation between pressure, force, and area ($p=F/A$). Neglecting the hydrostatic pressure from the injection fluid, if the force applied to the syringe is the same then it is the area of the bottom that affects the pressure. The larger the area, the lower the pressure. The staff should choose syringe B. The mathematical subject area is identified as algebra, such as to work with formulas and proportionality.	The assignment is to choose which syringe that gives the minimum pressure and to provide an argument for this choice. Only the force is given as a variable, and this is represented with a letter. Key words for the students can be *force* and *pressure*. The situation is illustrated with a figure in which it appears that syringe B has a greater diameter than syringe A.
Task 4	To derive a solution, the forces acting on the upper ball must be considered. Because it is levitating freely, it is in equilibrium and, according to Newton's first law, the net force on the ball is zero. The forces acting on the ball are the downward gravitational force, $F = mg$, and the upwards electrostatic force from the ball below, $F = kQ_1Q_2/r^2$. Setting these expressions equal to each other and solving for Q_1 (and assuming that $Q_1 = Q_2$) will give the charges asked for. The mathematical subject area is identified as algebra, such as to work with formulas and to solve quadratic equations.	The assignment is to calculate the charges on the balls. The mass of the balls and the distance between their centres are mathematical objects given numerically and explicitly in the assignment. The information about the charges' equal magnitude is textual and is a part of the description of the situation. There is also a figure of the balls on the thread illustrating the experiment.

Figure 3a. The first two steps in the analysis procedure for tasks 1 to 4.

III. Analysis of the textbooks and handbook – Answers and solutions	**IV. Argumentation for the requirement of reasoning**
Handbook: Formulas for power, P=ΔW/Δt, with the explanation "ΔW = the change in energy during time Δt"; for "work during lift", W_l = mg·h, with the explanatory text, "A body with weight mg is lifted to a height h. The lifting work is…"; and for potential energy with the text "A body with mass m at a height h over the zero level has the potential energy W_p = mg·h". **Mathematics book[3]:** Numerous examples and exercises of how to use formulas, e.g. on pages 28-30. **Physics book[4]:** Power is presented as work divided by time, and in on example work is exemplified as lifting a barbell. An identical example is found on page 130. An example of calculating work during a lift in relation to change in potential energy is found on page 136. Exercises 5.05 and 5.10 are solved by a similar algorithm.	The analysis of the textbooks shows that there are more than three tasks similar to the task being categorised with respect to the task variables, and these tasks can be solved with a similar algorithm. As mentioned in the method section, if the students have seen tasks solvable with a similar algorithm at least three times, it is assumed that they will remember the solution procedure. This task is then categorised as solvable using IR, in this case FAR.
Not a step to consider as this task is categorized as NMR.	Not a step to consider as this task is categorized as NMR.
Handbook: The relation p=F/A is defined. **Mathematics book:** Proportionalities are discussed and exemplified but are not used for general comparisons. **Physics book:** One example about how different areas affect the pressure and one exercise that is solved in a similar way by using a general comparison between different areas and pressure.	There is only one example and one exercise that can be considered similar with regard to the task variables and the solution algorithm. The formula is in the handbook, but there has to be some understanding of the intrinsic properties in order to be able to use the formula in the solution. This task is, therefore, considered to require some CR, in this case GCR, in order to be solved.
Handbook: Coulomb's law, $F = k \cdot Q_1Q_2/r^2$, with explanation "r = distance between the charges and … k =… ≈ $8.99 \cdot 10^9$ $Nm^2/(As)^2$". **Mathematics book:** Numerous examples and exercises of how to use formulas, e.g. on pages 28-30, and of solving quadratic equations on page 269. **Physics book:** Coulomb's law is introduced and exemplified, and there are at least three exercises of calculating the charge on different objects using this law. One example is of a levitating charge (p. 227), but in this case in a homogeneous electrical field instead of due to the electrostatic force from another charged particle. Two exercises of similar situations as in the example. Newton´s first law is formulated in the theory text (p. 91) where it is shown that the net force has to be zero if an object for example is at rest, and this relation is used on several different occasions in the book. The gravitational force is introduced on pages 92 and is then used throughout the book.	Considering the mathematical reasoning, there are more than three examples or exercises in the textbooks where the same algorithm has been used, i.e. to put two expressions equal to each other and then solve for one unknown variable, including taking the square root. However, there are not three or more examples considering the physics context. To solve the task, the student must first identify the force situation in order to know which expressions to equate. After having discussed this task in the reference group, it was concluded that analysing the physics context is not a part of the mathematical reasoning. Although mathematical reasoning is necessary to be able to solve the task, it is not sufficient, and although the mathematical reasoning can be considered as some kind of algorithmic, the task was categorised as requiring LCR, where the minor step is to analyse the physics.

Figure 3b. The last two steps in the analysis procedure for tasks 1 to 4.

	I. Analysis of the assessment task- Answers and solutions	II. Analysis of the assessment task- Task variables
Task 5	This task can be solved using the equilibrium of torque (moment of force), M = Fr, and the knowledge that the torque with respect to Anton must have the same magnitude as the torque with respect to Lars. The forces that act on the seesaw are of the same magnitudes as the gravitational forces, F= mg, on Lars and Anton, respectively. Assuming that Anton is placed 1.60 m from the rotation axis, one gets the equation $F_{Lars} \cdot r = F_{Aanton} \cdot 1.60$, which will give the position Lars must be in when the equation is solved. As in the examples above, the mathematical subject area was identified as algebra, more specifically to work with formulas and equations. A seesaw is a real-life situation often used as an example in mechanics and, therefore, was included in the analysis.	The assignment is to show where on the seesaw Anton and Lars can sit when it is in equilibrium. Mathematical objects that are given numerically in the assignment are the masses of Anton and Lars. In addition, the total length of the seesaw is given and there is a picture of a seesaw without any people on it
Task 6	To solve this task, the students are supposed to refer to Archimedes' principle. The greater the volume of the body under the water, the lager the buoyant force from the water. Assuming the body is in equilibrium at each step, the larger the buoyant force becomes the smaller the normal force from the stones becomes and thus there is less pressure from the stones. Therefore, it hurts less when the water level reaches higher on the body. This relation can be argued for using the formulas for Archimedes' principle, formulas for pressure, and the equilibrium of forces. The mathematical area could then be considered to involve formulas and proportionality. Following the solution proposal and the scoring rubric provided with the test, however, there is no need to use any mathematical relations or formulas to argue for the answer.	The assignment is to explain why it does not hurt as much when you are in deeper water. No mathematical objects are given explicitly in the task. The situation refers to a real-life event of walking in water. Bathing is a common situation referred to when discussing Archimedes' principle. The depth of the water is also indicated in the assignment as important.

Figure 3c. The first two steps in the analysis procedure for tasks 5 and 6.

III. Analysis of the textbooks and handbook – Answers and solutions	IV. Argumentation for the requirement of reasoning
Handbook: Formula for Torque, M=Fr, with explanatory text "r is the perpendicular distance from the rotation axis to the line of action of the force. At equilibrium $\sum$ Fr = $\sum$ M = 0" together with a figure of M around a rotation axis with F and r marked. **Mathematics book:** Numerous examples and exercises on how to use formulas (e.g. on pages 28-30) and how to solve equations. **Physics book:** The relation for torque is formulated with words in the theory text. When introducing torque, the theory also refers to a seesaw both in text and with images (p. 105). Two examples use the formula for torque as defined in the handbook. One of the examples is similar to this task except that one does not have to assume any distance. There are some exercises using a similar algorithm, but these are for calculating masses (via force) from given distances instead of distances from given masses.	The algorithmic procedure to solve a task involving a seesaw has been seen both in the theory text and in the examples. There are plenty of exercises for how to handle expressions and solve equations with one unknown variable. The difference in this case is that none of the distances are given in the task. There are, therefore, two unknown variables in the expression, and one of the distances has to be assumed, by using the information about the total length of the seesaw. After discussion about this task, it is categorised as requiring LCR. The minor step in this case is to realise that one has to make an assumption of one of the distances in order to be able to solve the task, and this is regarded as demanding some intrinsic mathematical understanding.
Handbook: Archimedes' principle is formulated with the words, "The buoyant force on an object is equal to the weight of the displaced fluid" that appear on the same page as the formula for pressure, p=F/A. **Mathematics book:** Numerous examples and exercises on how to use formulas, e.g. on pages 28-30, and exercises on proportionality on pages 73 and 75, but these are not used for general comparison. **Physics book:** Archimedes' principle is formulated with words and as an expression (p. 171), and there is one example that relates volume to the buoyant force.	Following the scoring rubric of what is demanded of a student to solve this task, there is no need to refer to the formulas or to use them to argue for the given explanation. The student needs to mention Archimedes' principle and that the buoyant force increases when the volume of the body in the water increases, but he/she does not need to explain why or show how the volume increase is related to the force increase. They also have to mention something about how this increased buoyant force decreases the normal force, but according to the scoring rubric there is no need to use the relation for pressure to show why this decreased normal force makes it hurt less. The space given to write the answer also indicates that a few lines are sufficient as an answer. After discussing this task and the minimum solution that is required of a student, it is decided that the reasoning is mainly physical and that mathematical reasoning is not necessary to solve this task. It is then categorised as solvable with NMR.

Figur 3d. The last two steps in the analysis procedure for tasks 5 and 6.

Results

The analysis showed that mathematical reasoning was required when solving physics tasks. Of the 209 analysed tasks, there were 76 % that required mathematical reasoning. The distribution of tasks categorised as requiring CR (CR-tasks) and tasks solvable with IR (IR-tasks) were a bit unbalanced. Of the tasks requiring mathematical reasoning, 46% were CR-tasks whereas the remaining ones were IR-tasks (Table 1).

Table 1

Categorisation results, overview

	Number of tasks	NMR %	CR %	IR %
Physics A	103	21	33	46
Physics B	106	26	38	36
Total	209	24	35	41

The result also showed some differences in the categorisation with respect to the Physics A and Physics B courses. There were slightly more NMR-tasks and CR-tasks in the Physics B tests than in the Physics A tests. A more distinct difference was seen among the IR-tasks, with the greater number of these tasks in Physics A tests (Table 1).

Table 2

Categorisation results, detailed.

	Number of tasks	NMR n	NMR %	FAR n	GAR n	IR %	LCR n	GCR n	CR %	GCR %	IR+LCR %
Physics A Dec98	20	3	15	6	6	60	4	1	25	5	80
Physics A May02	20	4	20	3	3	30	5	5	50	25	55
Physics A Dec04	19	4	21	7	1	42	2	5	37	26	53
Physics A May05	19	5	26	6	2	42	4	2	32	11	63
Physics A Dec08	25	6	24	12	1	52	4	2	24	8	68
Total Physics A	103	22	21	34	13	46	19	15	33	15	64
Physics B May02	18	2	11	7	0	39	5	4	50	22	67
Physics B May03	19	5	26	8	1	47	3	2	26	11	63
Physics B May05	23	7	30	4	3	30	5	4	39	17	52
Physics B Feb06	23	10	43	8	0	35	2	3	22	13	43
Physics B April10	23	4	17	5	2	30	4	8	52	35	48
Total Physics B	106	28	26	32	6	36	19	21	38	20	54
Total	209	50	24	66	19	41	38	36	35	17	59

A majority of the IR-tasks (78%) were solvable with FAR and the rest were solvable with GAR. The CR-tasks were separated into LCR and GCR. In general, Physics B tests consisted of more GCR-tasks than Physics A tests, and the amount of LCR-tasks was almost the same (Table 2). When comparing tests from different years, the analysis showed a notable variation in the proportions of the different mathematical reasoning types. There was no consistency among the tests with respect to this analysis (Table 2).

Discussion and implications

The national tests are used in the present study to represent the mathematical reasoning that is required to master or fully master the physics courses according to the syllabuses and curriculum. Because of the way the national physics tests are constructed, students that have fully mastered the physics curricula should have the ability to solve any of the tests for the related course. The fact that slightly less than one-third of the tasks on some of the 10 tests in this study require CR (Table 2) does not weaken the overall result that CR is significant for fully mastering the physics curricula.

The fact that a majority of the tasks require mathematical reasoning shows that the ability to reason mathematically is an important competence and an integral part when taking physics tests. Mathematical reasoning is defined as a process to reach conclusions when solving tasks. When students have the ability to use creative mathematical reasoning, they know how to argue and justify their conclusions and they can draw on previous knowledge. As it is not enough to only use IR to solve a majority of the tasks in a test, but especially CR is required, a creative mathematical reasoning competency can be regarded decisive when students develop their physics knowledge. At first glance, it might be reasonable to assume that CR is required to get a higher grade on a test, and this hypothesis was tested in a follow-up study (Author, 2013). It was shown in that study that in order to get one of the higher grades students had to solve tasks requiring CR in five out of eight national physics tests. For the three tests not requiring CR, students' actual results on these three tests were compared to which tasks they had solved, and it was concluded that even though it was possible to get a higher grade without using CR, this rarely occurred.

The conclusion that CR is vital to students' development of physics knowledge is based on that the Swedish national physics tests are a concretisation of the goals in the syllabuses and in the curriculum of what should have been achieved after completing the physics courses. The goals and the subject descriptions in the Swedish syllabuses and curriculum of what it means to know physics are quite rich and highly in accordance with the content and cognitive domains in the TIMSS Assessment framework (Garden et al. 2006; SNAE, 2009b). Although this study deals with the Swedish settings, the alignment with TIMSS suggests that these results are relevant to an international context.

As mentioned above in the section "Learning physics", individuals' understanding of the relevance of different concepts in various contexts has to be examined in order to discuss what has been learned. The present study does not claim anything about individual students' learning. However, it is shown that mathematical reasoning in general and CR in particular is vital when students solve tasks in physics. Since CR is based on an intrinsic understanding of a concept and the ability to use the concept in novel situations, this is in line with diSessa's (2004) view of learning as a development of the ability to use a concept in shifting contexts.

This study is situated within the "Mathematics in Physics" research field (see Introduction). The literature suggests how mathematical knowledge influences the learning of physics and the importance of understanding how mathematics is used in physics. From the results in this study, mathematical reasoning can be concluded to be a central aspect of this mathematical knowledge. In particular CR is decisive to fully master the physics curricula. To achieve this CR competency, students must be provided opportunities to develop and practice creative mathematical reasoning. This could take place both in the physics classes and in the mathematics classes. According to references discussed in the Introduction as well as in the Method section, it is common that students in physics classes solve routine tasks and focus on manipulations of formulas instead of focusing on the conceptual understanding. Similar conclusions are drawn regarding the mathematics classes; it is found that the focus is on algorithmic procedures and no extensive opportunities to develop different kinds of CR are provided (e.g. Boesen et al., 2014).

It is known that tests have an indirect role for students learning, both as formative, when students get feedback on their solutions, and as summative, when the character of the tasks give students indications of what competences that are sufficient for handling mathematical tasks. Analyses of teacher-made mathematics tests have shown that these focused largely on IR, in contrast to the national mathematics tests, which had a large proportion of tasks requiring CR (Palm et al., 2011). In view of the result of Boesen et al., about the situation in the mathematics classes, and of Palm et al., about teacher-made mathematics tests, it is reasonable to assume that the teacher-made tests represent respective mathematics teacher's practice. This assumption is further supported by one of the results in Boesen (2006), where teachers indirectly claim that their assessments align with the instructional practice. In the same way it is assumed that physics teachers' practice are reflected in the physics tests they construct. As discussed above, the classroom situations in physics and mathematics can be considered similar. Thus, a reasonable conclusion is that there is a similar discrepancy regarding physics tests, i.e. that there is a larger proportion of CR in the national physics tests than there is in the teacher-made tests.

From the discussion above it seems that although the intense efforts that have been made to change practice through policy changes, discussed in the "Mathematics in the syllabuses" section, students are provided limited opportunities to develop the creative mathematical reasoning competency that is required to fully master the physics curricula. It can be assumed that the implementation work of the new curricula in school, concretised through national tests, has not worked as intended. The importance of the relation between mathematics and physics has been known for a long time. What has been found in this study is that the ability to mathematically argue and reason is decisive in order to fully master the physics curricula, and this should have implications on how the education is organised and carried out.

Notes

[1] Author's translation

[2] Originally called *creative mathematical founded reasoning*.

[3] The mathematics text-book in all examples is Björk & Brolin (2001)

[4] The physics text-book in all examples is Pålsgård et al. (2005a)

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